LEAST MEAN SQUARE, FINITE LENGTH, PREDICTIVE DIGITAL FILTERS

E. H. Satorius and J. R. Zeidler

Naval Undersea Center, San Diego, California 92132

Abstract

Useful analytic solutions of the discrete Wiener-Hopf (W-H) matrix equation (using the method of undetermined coefficients (UC)) are presented for general rational input spectra. For the specific case of N sinusoids in white noise, the W-H equation is transformed via the UC method into a set of N coupled linear equations which decouple as the filter length becomes large. For all pole input spectra, the W-H equation transforms into a pxp Vandermonde matrix equation where p is the number of poles in the input spectra.

Introduction

In recent years, considerable interest has been focused on linear predictive filtering. Makhoul [1] has summarized many of its applications to such areas as neurophysics, geophysics, speech communications, etc. The basic steps in linear predictive filtering consists of first obtaining the autocorrelation lags, $\phi_{xx}(\ell)$, of a process, $x(k)$. We will assume that the process is stationary and, therefore, the lags are given by: $\phi_{xx}(\ell) = E[x(k)\cdot x(k+\ell)]$ where $E[\cdot]$ denotes expectation. The next step is to obtain the optimum linear predictive filter, $w^*(k)$ for $k = 0,1,\ldots,L-1$ from the discrete Wiener-Hopf matrix equation:

$$\underline{\underline{R}}\cdot\underline{W}^* = \underline{P}. \tag{1}$$

In (1), $\underline{\underline{R}}$ is the LxL autocorrelation matrix with elements $(\underline{\underline{R}})_{\ell k} = \phi_{xx}(\ell-k)$; $\underline{W}^*$ is the L-dimensional optimal weight vector (column vector) with elements $(\underline{W}^*)_k = w^*(k-1)$; and $\underline{P}$ is the L-dimensional cross-correlation column vector with elements $(\underline{P})_k = \phi_{xx}(k+\lambda-1)$ where λ is a positive integer representing the prediction distance of the filter.

In this paper, we will present analytic solutions of eq.(1) when the autocorrelation lags are known exactly. In particular, we will consider the solution of (1) when the input spectral density,

$$S_{xx}(z) \equiv \sum_{k=-\infty}^{\infty} \phi_{xx}(k)\ z^{-k},$$

is a rational function of z. An analytic solution for this case is particularly interesting since it displays the end effects of the finite length filter and shows when these effects become negligible. These results are especially relevant to the all pole modeling of spectra which contain both poles and zeroes. The approach which is used in this paper to obtain $w^*(k)$ is based on the method of undetermined coefficients. (A similar method was applied to the continuous analog of (1) by Zadeh and Ragazzini [2]).

Analysis

We shall write the spectral density, $S_{xx}(z)$, as follows:

$$S_{xx}(z) = A\,\frac{\prod_{m=1}^{M}(z-b_m)(z^{-1}-\bar{b}_m)}{\prod_{n=1}^{N}(z-a_n)(z^{-1}-\bar{a}_n)} \tag{2}$$

where $|a_n| < 1$; $|b_m| < 1$; and $\bar{a}$ denotes the complex conjugate of a. The zeroes and poles of $S_{xx}(z)$ occur at $\{b_m, \bar{b}_m^{-1}\}$ and $\{a_n, \bar{a}_n^{-1}\}$ respectively. The expansion in (2) represents the spectral density of a general complex process. For a real process, the a_n as well as the b_m will always occur in complex conjugate pairs. The autocorrelation function corresponding to (2) is given by (assuming N>M):

$$\phi_{xx}(k) \equiv \frac{1}{2\pi j}\oint_{|z|=1} S_{xx}(z)\ z^{k-1}\,dz \qquad k=0,\pm1,\ldots$$

$$= \sum_{n=1}^{N} D_n(k)\ e^{-\alpha_n|k|+j\omega_n k}, \tag{3}$$

where $a_n \equiv \exp(-\alpha_n+j\omega_n), (\alpha_n>0)$; and $D_n(k)$ is given by:

$$D_n(k) \equiv A_n = \frac{B\,a_n^{N-M-1}}{a_n - \bar{a}_n^{-1}}\ \frac{\prod_{m=1}^{M}(a_n-b_m)(a_n-\bar{b}_m^{-1})}{\prod_{\substack{\ell=1\\ \ell\neq n}}^{N}(a_n-a_\ell)(a_n-\bar{a}_\ell^{-1})}\ ;\ k\geq 0,$$

and, $D_n(k) = \bar{A}_n$ for $k < 0$, where:

$$B = (-1)^{M-N}\left(\prod_{m=1}^{M}\bar{b}_m \Big/ \prod_{n=1}^{N}\bar{a}_n\right)\cdot A.$$

The discrete Wiener-Hopf matrix equation (1) can be written in component form as follows:

$$\sum_{k=0}^{L-1} \phi_{xx}(\ell-k)\, w^*(k) = \phi_{xx}(\ell+\lambda) \qquad (4)$$

$$\ell = 0,1,\ldots,L-1$$

Eq.(4) will be solved by the method of undetermined coefficients. In this method, an assumed solution for $w^*(k)$ in terms of unknown constants is substituted into (4). The assumed solution (in analogy with the continuous problem [2]) is given as follows:

$$w^*(k) = \sum_{m=1}^{M} \{B_m^+ e^{-\mu_m k+j\theta_m k} + B_m^- e^{-\mu_m(L-1-k)+j\theta_m k}\}$$

$$+ \sum_{r=1}^{N-M} \{C_r^+ \delta(k-r-1) + C_r^- \delta(k+r-L)\} \qquad (5)$$

$$k = 0,1,\ldots,L-1$$

where $b_m \equiv \exp(-\mu_m+j\theta_m)$, $(\mu_m>0)$ and $\delta(k)$ is the Kronecker delta function, i.e., $\delta(k)=0$ for $k\neq 0$ and $\delta(0)=1$. In writing (5) we have assumed (as we will throughout the rest of this paper) that $L> N-M$. Substituting (5) into (4) with $\phi_{xx}(\ell)$ given by (3) yields the following equations for the undetermined constants:

$$\sum_{m=1}^{M} \{\frac{B_m^+}{1-e^{\alpha_n-\mu_m-j(\omega_n-\theta_m)}} + \frac{B_m^- e^{-\mu_m(L-1)}}{1-e^{\alpha_n+\mu_m-j(\omega_n-\theta_m)}}\}$$

$$+ \sum_{r=1}^{N-M} C_r^+ e^{\alpha_n(r-1)-j\omega_n(r-1)} = e^{-\alpha_n\lambda+j\omega_n\lambda} \qquad (6)$$

$$n=1,\ldots,N$$

and,

$$\sum_{m=1}^{M} \{\frac{B_m^+ e^{-\mu_m L+j\theta_m L}}{1-e^{-\alpha_n-\mu_m-j(\omega_n-\theta_m)}} + \frac{B_m^- e^{\mu_m+j\theta_m L}}{1-e^{-\alpha_n+\mu_m-j(\omega_n-\theta_m)}}\}$$

$$- \sum_{r=1}^{N-M} C_r^- e^{\alpha_n r+j\omega_n r} = 0 \qquad (7)$$

$$n=1,\ldots,N.$$

Eq.(5) together with eqns.(6) and (7) give the expression for the impulse response of the finite length, optimum predictive digital filter, $w^*(k)$. (A more complete derivation of these equations will be presented in a forthcoming paper). A number of interesting properties of $w^*(k)$ can be seen from these equations. First, it is seen from eq.(5) that $w^*(k)$ consists of sums of damped exponentials, $\exp(\pm\mu_m k+j\theta_m k)$ as well as impulses. It is further seen that the exponentials, which are the zeroes of the input spectral density, decay away from each end of the filter with the B_m^+ and B_m^- representing the amplitudes of the damped exponentials which decay away from the beginning and end of the filter, respectively. Likewise, the C_r^+ and C_r^- represent the amplitudes of the impulses which occur at the beginning and end of the filter, respectively. Therefore, the constants associated with the "-" superscripts can be thought of as reflection amplitudes which are the direct result of the finite filter length. From eqn.(6) it is seen that the B_m^- couple into the N equations for the B_m^+ and C_r^+ through coupling coefficients which are proportional to $\exp(-\mu_m L)$. Vice versa, from eqn.(7) it is seen that the B_m^+ couple into the N equations for the B_m^- and C_r^- (the reflection amplitudes) through coupling coefficients which are also proportional to $\exp(-\mu_m L)$. As $L\to\infty$, these coupling coefficients approach zero and from eqn. (7), the reflection amplitudes, B_m^- and C_r^- also approach zero. Therefore, as $L\to\infty$, i.e., as $\exp(-\mu_m \cdot L)\to 0$, the reflection amplitudes approach zero and $w^*(k)$ can be represented as a sum of damped exponentials and impulses all of which occur at the beginning of the filter. It should be noted that the reflection amplitudes, B_m^- and C_r^-, are similar to the reflection coefficients which appear in the Durbin algorithm (see,e.g.,[1,Sect.2]).

Special Cases

We now wish to apply the above method of undetermined coefficients to two special cases of practical interest. The first case considered is that of all pole input spectra,i.e., when $S_{xx}(z)$ is given as follows:

$$S_{xx}(z) = \frac{1}{\prod_{n=1}^{N} (z-a_n)(z^{-1}-\bar{a}_n)} \qquad (8)$$

For this case there are no zeroes, and the expression for $w^*(k)$ is given by (5) with M=0 and $C_r^- = 0$ viz.,

$$w^*(k) = \sum_{r=1}^{N} C_r^+ \delta(k-r-1), \qquad (9)$$

where the C_r^+ satisfy:

$$\sum_{r=1}^{N} C_r^+ e^{(\alpha_n-j\omega_n)(r-1)} = e^{-\alpha_n\lambda+j\omega_n\lambda} \qquad (10)$$

$$n=1,2,\ldots,N$$

Note that equations (9) and (10) represent a transformation of the original LxL equation for $w^*(k)$ (eq.(4)) into an NxN Vandermonde matrix equation (eq.(10)). Explicit relations for the C_r^+ may be obtained from (10) by making use of the closed form expressions which have been derived for the inverse of a Vandermonde matrix (see,e.g., [3] , [4, pp.220-223]).

The second special case considered is that of inputs which consists of multiple sinusoids in white noise, i.e., when $\phi_{xx}(k)$ is of the following form:

$$\phi_{xx}(k) = \sigma_o^2\, \delta(k) + \sum_{n=1}^{N} \sigma_n^2\, e^{j\omega_n k}, \qquad (11)$$

where σ_o^2 is the noise power; σ_n^2 is the power of the n^{th} sinusoid; and the ω_n represent the frequencies of the sinusoids. Eq.(11) is a limiting case of the more general autocorrelation function given by (3). For this special case, it can be seen that w*(k) given by:

$$w^*(k) = \sum_{n=1}^{N} A_n e^{j\omega_n k}, \tag{12}$$

will lead to a solution of (4). Substitution of (12) into eq.(4) with $\phi_{xx}(k)$ given by (11) and equating coefficients of $\exp(j\omega_r \ell)$ leads to the following N equations in the N constants, A_r:

$$A_r + \sum_{\substack{n=1 \\ n\neq r}}^{N} \gamma_{rn} A_n = \frac{e^{j\omega_r \lambda}}{L+\sigma_o^2/\sigma_r^2} \tag{13}$$

$$r=1,2,\ldots,N$$

where in (13), γ_{rn} is given by:

$$\gamma_{rn} = \frac{1}{L+\sigma_o^2/\sigma_r^2} \frac{1-e^{j(\omega_n-\omega_r)L}}{1-e^{j(\omega_n-\omega_r)}} \tag{14}$$

The solution of eq.(13) for the A_n completely determines w*(k) through eq.(12). Equations (12)-(14) represent a transformation of the LxL equation (4) into an NxN equation (eq.(13)). Eq.(12) implies that when the input consists of N sinusoids in white noise, the optimal linear predictive filter, w*(k) can be expressed as a sum of N sinusoids at the same frequencies as the input signals. From eq.(13) it is seen that the amplitudes, A_n, of these sinusoids are all coupled together through coupling coefficients, γ_{rn}, which approach zero as $L\to\infty$ (as can be seen from (14)). Therefore, as $L\to\infty$, the $\gamma_{rn}\to 0$, and the A_n are given to a good approximation by:

$$A_n = \frac{e^{j\omega_n \lambda}}{L+\sigma_0^2/\sigma_n^2} \tag{15}$$

The application of equations (12)-(15) to the adaptive least mean squares filtering of multiple sinusoids in white noise will be considered in a forthcoming paper.

Conclusion

Solutions of the discrete Wiener-Hopf matrix equation for the optimum linear predictive filter have been presented for general rational input spectra using the method of undetermined coefficients. This method was also applied to the special cases of all pole input spectra and inputs consisting of multiple sinusoids in white noise.

References

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