LINEAR PREDICTION AND MAXIMUM ENTROPY SPECTRAL ANALYSIS OF FINITE BANDWIDTH SIGNALS IN NOISE

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ABSTRACT

The optimal minimum-mean-square-error discrete filters are obtained for finite bandwidth signals in noise represented by a rational pole-zero model in the spectral domain. An analytical technique called the method of undetermined coefficients is used to solve the Wiener equation for the optimal prediction coefficients. These coefficients are then used to examine (a) the frequency response of the Wiener prediction filter, and (b) the maximum entropy spectral estimator. Special cases considered include the limiting case of narrow bandwidth signals and large prediction filter lengths.

I. Introduction

Recently the use of linear prediction filters (LPF) for enhancing the detectability of narrowband signals in broadband noise has been considered [1-6]. For the case of sinusoidal signals in uncorrelated noise it has been shown in [1-6] that an LPF forms narrow bandpass filters around the sinusoids. The widths of these filters is inversely proportional to the length L of the LPF and approach zero as L becomes very large. It is the purpose of this paper to extend the results in references [1-6] to the case of signals with non-zero bandwidth. An example could be that of a stationary bandpass signal process corrupted by additive noise. Also, we will consider the closely related problem of obtaining spectral estimates by the maximum entropy method (MEM) for the case of finite bandwidth signals in additive noise. The approach used will be to first develop a pole-zero model for the signal plus noise spectrum. Then the normal equations for the LPF coefficients will be solved by the method of undetermined coefficients.

II. Determination of Appropriate Pole-Zero Models for Finite Bandwidth Signals in Additive Noise

In this paper it will be assumed that the noise background is uncorrelated, although the extension of the pole-zero models presented here to the correlated noise case is straightforward. A reasonably simple model for an input process consisting of N finite bandwidth signals in uncorrelated noise may be represented by the autocorrelation function \( r_{xx}(k) \) where \( x(t) \) is the time series of the input process:

\[
\begin{align*}
  \frac{1}{2\pi} \int H(z)H(z^{-1}) z^{-k-1} \, dz &= A^2, \\
  r_{xx}(k) &= \sigma^2_0 + \sum_{n=1}^{N} \sigma^2_n \exp(-\alpha_n |k|) \cos \omega_n k 
\end{align*}
\]

In (1), \( \sigma^2_0 \) represents the white noise power; \( \sigma^2_n, \alpha_n, \) and \( \omega_n \) represent the power, bandwidth, and frequency, respectively, of the \( n \)-th signal; and \( \delta(k) \) is the Kronecker delta function.

The model represented by (1) is a mixed autoregressive-moving average (ARMA) model with \( 2N \) AR terms and \( 2N \) MA terms. Equivalently, this model can be represented by the transfer function

\[
H(z) = \prod_{n=1}^{N} \frac{\left( z - e^{-\alpha_n \pm j\omega_n}\right)}{\left( z - e^{-\alpha_n \mp j\omega_n}\right)},
\]

where \( A \) is a real constant. The zeroes of (2) may be obtained by factoring the power spectral density \( S_x(\omega) \) which is the transform of (1). In general, such a factorization of the spectrum is analytically tractable only for small values of \( N \) and does not provide much insight into the analytical structure of the zeroes of (2). However, when the \( \alpha_n \) in (1) are all much smaller than unity so that the background noise spectral density may be closely approximated from \( S_x(\omega) \) between the signal frequencies (i.e., no appreciable overlap of signal spectra), then a simple approximating expression for the zeroes in (2) may be derived.

The assumption that the background noise may be closely approximated from \( S_x(\omega) \) is consistent with requiring that the poles and zeroes of \( H(z) \) occur in pairs and that the distance between each member of the pair is very small. Therefore, the product \( H(z)H(z^{-1}) \) will be essentially constant except near a pole-zero pair, i.e.,

\[
H(z)H(z^{-1}) = A^2, \quad \text{for } z \neq e^{-\alpha_n \pm j\omega_n},
\]

(3)

We can now obtain a simple expression for the zeroes of \( H(z) \) by comparing \( r_{xx}(k) \) evaluated from

\[
\begin{align*}
  r_{xx}(k) &= \frac{1}{2\pi} \int H(z)H(z^{-1}) z^{-k-1} \, dz
\end{align*}
\]
using (3) and with \( r_{\infty}(k) \) given by (1). The expression for the zero locations of \( H(z) \) is then given by

\[
\exp(-\mu_n + it_n), \quad n = 1, \ldots, 2N
\]

where\n
\[
\mu_n = \sum_{n=n}^{\infty} |\omega_n|^2 + \text{SNR}_{\infty} \omega_n\]

and \( \mu_{n+N} = \mu_n \) for \( n = 1, \ldots, N \) \( (5a) \)

\[
\theta_n = \omega_n \quad \text{and} \quad \theta_{n+N} = \omega_n \quad \text{for} \quad n = 1, \ldots, N \quad (5b)
\]

In (5), \( \text{SNR}_{\infty} = \sigma^2 / \sigma_0^2 \) and is the SNR of the n-th signal. Equations (5a) and (5b) show that for small signal bandwidths (on the order of one percent of Nyquist) the zeroes of \( H(z) \) are displaced slightly back along radial lines from the signal poles locations. As the \( \omega \to 0 \), both the zeroes and the poles approach the unit circle. This implies that in the limit of zero signal bandwidths (i.e., sinusoids in white noise) the appropriate time series model for the process represented by (1) is an ARMA model of order \( (2N, 2M) \) with identical AR and MA terms. This limiting result was also derived using a difference equation approach in [7].

III. The Linear Prediction Filter Coefficients

In this section we will derive the LPF coefficients, \( w^*(k) \), for \( k = 0, \ldots, L-1 \). The autocorrelation lags will be assumed to be known exactly and be given by (1). This paper will not address the problems inherent in estimating the autocorrelation lags. It will also be assumed that the \( \alpha_n \) are small enough such that (5a) and (5b) are valid. The \( w^*(k) \) satisfy the discrete Wiener-Hopf equation:

\[
L-1 \sum_{k=0}^{L-1} r_{xx}(p-k) w^*(k) = r_{xx}(p+\Delta), \quad p=0, \ldots, L-1 \quad (6)
\]

where \( \Delta \) is the prediction distance. Equation (6) may be solved using discrete Wiener filter theory [8]. When the power spectral density is a rational function of \( z \), an expression for \( w^*(k) \) using the method of undetermined coefficients is given in [9] in terms of the poles and zeroes of \( H(z) \). Recent applications of the method of undetermined coefficients has appeared in the area of multiple sinusoid enhancement [10] and MEM analysis of sinusoids in lowpass noise [13]. For the special case when \( H(z) \) is given by (2) and the zeroes are given by (5), the expression for \( w^*(k) \) becomes

\[
w^*(k) = \sum_{n=1}^{2N} \left( \frac{B^+}{B^-} e^{-\mu_n + j\omega_n k} \right) w_n^*(L-k) + \sum_{n=1}^{2N} \left( \frac{B^-}{B^+} e^{\mu_n + j\omega_n k} \right) w_n^*(L-k-1) \quad (7)
\]

where \( w_{n+N} = -w_n \) for \( n = 1, \ldots, N \); \( w_n \) is given by (5) and the \( B^+ \) are obtained from:

\[
2N \sum_{n=1}^{2N} \frac{B^+}{B^-} e^{-\mu_n + j\omega_n k} \left( \frac{B^+}{B^-} e^{-\mu_n + j\omega_n k} \right) + \frac{B^-}{B^+} e^{\mu_n + j\omega_n k} \left( \frac{B^-}{B^+} e^{\mu_n + j\omega_n k} \right) = 0, \quad n = 1, \ldots, 2N \quad (8)
\]

where in (8) and (9) \( a_{n+N} = a_n \) for \( n = 1, \ldots, N \).

Equation (7) shows that the LPF coefficients can be expressed as a sum of damped exponentials which decay away from each end of the filter. The \( B^\pm \) represent the amplitudes of the damped exponentials and may be obtained through the set of coupled equations given by (8) and (9).

To gain more insight into the structure of the LPF coefficients we will neglect the interaction between different signal components which is manifested in (8) and (9) and will therefore examine the LPF coefficients for one complex signal of finite bandwidth in white noise. The complex autocorrelation function for this case is given by

\[
r_{xx}(k) = \sigma^2 \delta(k) + \frac{1}{2} \sigma_0^2 e^{-\alpha_1 |k| + j\omega_1 k} \quad (10)
\]

The expression for the LPF coefficients is given by

\[
w^*(k) = B^+ e^{-\mu_1 k + j\omega_1 k} \quad \text{and} \quad \frac{B^-}{B^+} e^{-\mu_1 k + j\omega_1 k} \quad (11)
\]

where \( \mu_1 = [a_1^2 + \text{SNR}_{\infty} a_1]^{1/2} \) and the \( B^\pm \) may be obtained from (8) and (9). Two important limiting cases may now be considered. First, as \( \alpha_1 \to 0 \), \( w^*(k) \) approaches

\[
w^*(k) = \frac{e^{j\omega_1 (k+\Delta)}}{L + \sigma_0^2 / \sigma_1^2} \quad (12)
\]

Equation (12) is, of course, identical to the results obtained in [1-5] for the case of one complex exponential signal in white noise. The second limiting case is when the filter length \( L \) is sufficiently long such that

\[
\mu_1 L >> 1 \quad (13)
\]

In this case the LPF length appears infinite to the input process and, as noted in [9], the reflection amplitude \( B_1^+ \) in (11) approaches zero. The resulting limiting expression for \( w^*(k) \) is given by

\[
w^*(k) = e^{-\mu_1 k + j\omega_1 k} \frac{\alpha_1 - a_1^2 e^{j\omega_1 k}}{(1 - e^{1 - 1}) e^{\mu_1 L + j\omega_1 k}} \quad (14)
\]
Transforming (14) gives the following expression for the frequency response $H^*(\omega)$ of the LPF:

$$H^*(\omega) = \sum_{k=0}^{L-1} e^{-j\omega k} a_k$$

For the correlation lags are given, it is seen from (15) that $Q(\omega)$ reduces to

$$Q(\omega) = \frac{1}{1 - e^{-2j\omega \Delta}}$$

Equation (15) implies that the LPF forms a bandpass filter around the signal and the limiting (for large $L$) bandwidth (i.e., the 3dB width of $|H^*(\omega)|^2$) of the LPF is given by $\omega_\Delta$. Therefore, for low SNR inputs can the LPF bandwidth reduce to the signal bandwidth as $L \to \infty$. For higher SNR values (SNR $\geq a_\Delta$) the filter passband will be increasingly wider than the signal bandwidth.

IV. Maximum Entropy Spectral Analysis of a Finite Bandwidth Signal in Noise

The MEM spectral estimate, $S(\omega)$, may be written as follows [10]:

$$S(\omega) = \sigma_n^2 Q(\omega)$$

where

$$Q(\omega) = \left| 1 - e^{-2j\omega \Delta} \right|^{-2}$$

and $\sigma_n^2$ is the minimum mean square error of the LPF and is given by

$$\sigma_n^2 = \sum_{k=0}^{L-1} \frac{e^{-2j\omega \Delta} - 1}{2j\omega \Delta} \left( e^{j\omega \Delta} - 1 \right)$$

In this section we will consider the case when the correlation lags are given by (16). Then $H^*(\omega)$ may be obtained from (11) with $\Delta = 1$:

$$H^*(\omega) = B_1^+ \left[ \frac{e^{-j\omega \Delta} - 1}{j\omega \Delta} \right] + B_1 e^{-j(\omega \Delta)}$$

The form of $S(\omega)$ in the limiting case of $a_\Delta \to 0$ has already been studied in detail by a number of authors [11,13]. In the limit when (13) is satisfied, it is seen from (15) that $Q(\omega)$ reduces to:

$$Q(\omega) = \frac{\cosh(a_\Delta) - \cos(\omega_\Delta \omega)}{\cosh(a_\Delta) - \cos(\omega \omega_\Delta)}$$

which is a scaled copy of the input power spectrum. Note that there are at least two ways in which (13) may be satisfied. The first way is for $L \to \infty$ in which case (20) becomes valid. This result is expected since only an infinite order AR model can model a finite order ARMA process [12]. The second way for (13) to be satisfied is for the SNR to become large. In this case the zeroes of $H(z)$ move closer to the origin along radials and the ARMA model represented by (2) can be approximated by an AR process. For this case the MEM spectral estimate, which assumes an AR model, provides a good approximation to the input spectrum.

V. Conclusion

In this paper we have examined the linear prediction filtering and MEM spectral analysis of finite bandwidth signals in white noise. The extension of the results presented here to the case of correlated additive noise is straightforward. For instance, when the signal bandwidths are very small, the zeroes would still be given by (5) for the correlated noise case, except with SNR in (5) replaced by SNR per unit bandwidth at the $n$-th signal frequency. Additional poles and zeroes of the noise power spectrum would also be included and the LPF coefficients could then be computed as in [9] and [13].

REFERENCES


