The Transversal and Lattice Multiple Modulus Algorithms for Blind Equalization of QAM Signals

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Abstract. The most popular algorithm for blind equalization of QAM signals is the constant modulus algorithm (CMA). However, because the CMA cost function is not matched to constellations with signal points on multiple radii, CMA can exhibit excessive misadjustment for QAM signals. This paper develops an alternative to CMA, the multiple modulus algorithm (MMA), which is based on stochastic gradient minimization of a cost function that goes to zero at each radius of the QAM constellation. For the applicable QAM constellations, MMA exhibits significantly reduced misadjustment compared to CMA. A lattice implementation of MMA is also presented.

1 Introduction

The constant modulus algorithm (CMA) [1], [2] is based on the minimization of the phase-blind cost function

\[ J_{\text{CMA}} = \frac{1}{4} E \left[ \left| \hat{a}(n) \right|^2 - R_2 \right]^2, \]

\[ R_2 = E \left[ |a_n|^4 \right] / E \left[ |a_n|^2 \right], \]

(1)

where \( \hat{a}(n) \) is the equalizer output. The expected values in the definition of the modulus constant \( R_2 \) are taken over the constellation points under the assumption that all points are equally probable. If, as in phase shift keying (PSK), all of the constellation points lie on a single radius \( r_2 = |a_n| \), then \( R_2 = r^2 \). In quadrature amplitude modulation (QAM), \( R_2 \) is related to an average squared amplitude of the symbols. In general, however, \( R_2 \neq E |a_n|^2 \) for QAM. Furthermore, if the QAM constellation points lie on the multiple radii \( r_i, i = 1, \ldots, \rho \), then, typically, \( R_2 \neq r_i^2 \), \( \forall i \leq \rho \). That is, \( R_2 \) does not go to zero at any of the signal points of most standard QAM constellations. The CMA stochastic gradient tap-weight update is [1], [2]

\[ w(n+1) = w(n) - \mu \Delta a(n)(|\hat{a}(n)|^2 - R_2)u(n) \]

(2)

where \( \mu \) is the adaptive step-size, \( \Delta \) indicates complex conjugate, \( w(n) = [w_0(n) \ w_1(n) \ \ldots \ w_{L-1}(n)]^T \) is the vector of L adaptive tap-weights and \( u(n) = [u(n) \ u(n-1) \ \ldots \ u(n-L+1)]^T \) is the vector of received signal samples.

This paper presents an alternative to CMA that is matched to specific QAM constellations. This alternative, namely the Multiple Modulus Algorithm (MMA), was originally proposed for real-valued pulse amplitude modulation (PAM) signals in [3]. The MMA cost function, generalized here for complex-valued signals, is

\[ J_{\text{MMA}} = \frac{1}{4} E \left[ \sum_{i=1}^{\rho} (|\hat{a}(n)|^2 - r_i^2)^2 \right]. \]

(3)

This paper is organized as follows. Section 2 presents the setup for consideration of the blind equalization problem. Section 3 presents the transversal and lattice MMA tap-weight updates and discusses the limitations of the applicability of the MMA approach. Section 4 contrasts the misadjustment of MMA with that of CMA via analytical arguments. Section 5 presents the results of simulation experiments that compare the convergence performance of transversal and lattice implementations of MMA and CMA. Section 6 presents the conclusions. The references are contained in Section 7 and the figures follow in Section 8.

2 Setup

The setup for consideration of the blind equalization problem in this paper is shown in Figure 1. Godard's [1] conditions on the transmitted signal \( a(n) \) are all assumed to be satisfied. The QAM constellations considered in this paper are all defined in Figure 5.21 (c) (8-QAM) and Figure 5.22 (16-, 32-, 64-, 128- and 256-QAM) on pp. 223-224 of [4].

The sequence \( a(n) \) passes through the channel \( h(n) \). The channel impulse response is \( h(n) = \sum_{k=1}^{3} h_k \delta(n-k) \), where

\[ h_k = \frac{1}{2\sqrt{h^T h}} \left[ 1 + \cos \left( \frac{2\pi}{W} (k-2) \right) \right], \]

\( k = 1, 2, 3 \)

(4)

and \( h = [h_1 \ h_2 \ h_3]^T \). The normalization yields \( \sum_{k=1}^{3} |h_k|^2 = 1 \) for all values of the bandwidth parameter \( W \). This channel model was used in [5] and it has since become a standard for the evaluation of the performance of adaptive algorithms in the face of variable amplitude distortion which can be controlled with the single parameter \( W \) [6], [7]. For an \( L=11 \) tap equalizer, the eigenvalue disparities of the autocorrelation matrix \( R = E[u(n)u^H(n)] \) for values of \( W = 2.9, 3.1, 3.3 \) and 3.5, are roughly equal to 7.8, 10.5, 13.4, and 16.8 dB, respectively. In order to focus this paper on the effects of the mismatch between the cost function and the constellation, it is assumed that the received signal \( u(n) \) is sampled at the symbol rate and that proper symbol synchronization and carrier recovery have been established. The real and imaginary parts of the complex-valued additive white Gaussian noise process \( v(n) \) are assumed to be independent and have equal variance \( \pm \sigma^2 \). When the equalizer is implemented with a finite impulse response (FIR) transversal filter, the symbol estimate at iteration \( n \) is given by the inner product

\[ \hat{a}(n) = w^H(n)u(n) = \sum_{\tau=0}^{L} w^*_\tau(n)u(n-\tau) \]

(5)
where $H$ indicates Hermitian transpose. Perfect equalization is achieved if the composite impulse response of the channel-equalizer combination is equal to a simple delay, i.e., $x(n) = [0 \cdots 0 1_d 0 \cdots 0]^T$ where $1_d$ indicates that $1$ occurs in the $d^{th}$ position. In this case, $\hat{a}(n) = a(n-d)$ and the equalizer perfectly estimates the symbol that was transmitted $d$ symbol times earlier.

3 The Multiple Modulus Algorithm (MMA)

The gradient of (3) is

$$\frac{\partial J_{\text{MMA}}}{\partial w(n)} = E \left[ \hat{a}(n) \left( \prod_{m=1}^{L} |\hat{a}(n)|^2 - r_1^2 \right) \left( \sum_{m=1}^{L} \prod_{m=1}^{L} |\hat{a}(n)|^2 - r_1^2 \right) \right] w(n). \quad (6)$$

Removing the statistical expected value operator from (6), the stochastic gradient update for a transversal implementation of MMA is given by

$$w(n+1) = w(n) - \mu \hat{a}(n) \left( \prod_{m=1}^{L} |\hat{a}(n)|^2 - r_1^2 \right) \left( \sum_{m=1}^{L} \prod_{m=1}^{L} |\hat{a}(n)|^2 - r_1^2 \right) w(n) \quad (7)$$

where $\mu$ is the adaptive step size. The error-function component of the tap-weight update term is given by

$$e_{\text{MMA}}(n) \triangleq \hat{a}(n) \left( \prod_{m=1}^{L} |\hat{a}(n)|^2 - r_1^2 \right) \left( \sum_{m=1}^{L} \prod_{m=1}^{L} |\hat{a}(n)|^2 - r_1^2 \right). \quad (8)$$

Since $|\hat{a}(n)|$ is a $(4p-1)^{th}$-order surface in $|\hat{a}|$, there is a practical limit to the number of radii to which the MMA approach can be applied and this is discussed further below. In contrast,

$$e_{\text{CMA}}(n) \triangleq |\hat{a}(n)|^2 - R_2 \quad (9)$$

is a $3^{rd}$-order surface for all modulations.

The development of MMA presented here is based on two modifications to the "standard" MMA error function $e_{\text{MMA}}(n)$ defined in (8). First, $e_{\text{MMA}}(n)$ is scaled such that $\max_{|\hat{a}| < r_p} e_{\text{MMA}}(\hat{a}) \leq \max_{|\hat{a}| < R_2} e_{\text{CMA}}(\hat{a})$ in the region $|\hat{a}| < r_p$. That is, $e_{\text{MMA}}(n)$ is scaled by the multiplicative constant

$$K \triangleq \max_{|\hat{a}| < r_p} e_{\text{CMA}}(\hat{a}) = \max_{|\hat{a}| < R_2} e_{\text{CMA}}(\hat{a}). \quad (10)$$

Second, in the region $|\hat{a}| > r_p$, the MMA error function is replaced with the CMA error function for a PSK modulation with amplitude $r_p$. The purpose of switching between error function components is to avoid the precipitous $(4p-1)^{th}$-order rise in the magnitude of (8) in the region $|\hat{a}| > r_p$. This second modification produces a more useful algorithm than the "standard" MMA because it increases the maximum value of $\mu$ that can be used, resulting in more rapid convergence. For example, in simulation experiments with 16-QAM, it was found that the second modification increased the maximum usable value of $\mu$ by a factor of 10.

With the modifications discussed above, the modified-MMA (M-MMA) tap-weight algorithm for complex-valued signals becomes

$$w(n+1) = w(n) - \mu e_{\text{M-MMA}}(n) w(n) \quad (11)$$

where

$$e_{\text{M-MMA}}(n) \triangleq \begin{cases} Ke_{\text{MMA}}(n), & |\hat{a}(n)| \leq r_p \\ e_{\text{CMA}}(n), & |\hat{a}(n)| > r_p \end{cases} \quad (12)$$

and

$$e_{\text{M-MMA}}(n) = K e_{\text{MMA}}(n), \quad |\hat{a}(n)| \leq r_p \quad (13)$$

$e_{\text{MMA}}(n)$ and $K$ are defined in (8) and (10), respectively. For simplicity in the remainder of this paper, we will refer to algorithm defined in (11)-(13) as (transversal) MMA.

Figures 2 (a)-(d) plot the magnitude of (12) as a function of radial distance from the origin for the 8-, 16-, 32- and 64-QAM constellations referred to in Section 2, respectively. In each case, the rightmost minimum (zero) corresponds to the outermost radius $r_p$ of the constellation. In parts (a) and (b) of Figure 2, the maxima of the error function component that occur between the origin and $r_p$ span a dynamic range that is slightly more than a factor of 2. Thus, within a factor of roughly 2, the information in the equalizer output regarding how well it is adjusted is equally weighted at all radii by the MMA error function for 8- and 16-QAM. For 32-QAM however, the span of the maxima within the outermost constellation radius covers roughly a factor 18. Thus, the information at a significant percentage of the radii in the constellation is under-valued by the 32-QAM MMA error function. For 64-QAM, the span of these maxima is greater than 6 orders of magnitude. Too much of the information in the equalizer output is ignored by the 64-QAM MMA error function. These observations foreshadow the results of our simulation experiments with MMA for these standard 8-, 16-, 32- and 64-QAM constellations. MMA as formulated in (11)-(13) was found to perform well for 8- and 16-QAM on the channel model described in Section 2. For 32-QAM, MMA converges to a solution that exhibits 32 distinct clusters in $\hat{a}(n)$, but the asymptotic gain of the equalizer is too low; the clusters occur on radii that are smaller than the true radii of the constellation. For 64-QAM, MMA simply fails to converge at all; no clusters corresponding to the true constellation points are ever observed in $\hat{a}(n)$. Therefore, for the remainder of this paper, our discussion of MMA will be limited to its application to these standard 8- and 16-QAM constellations.

The stochastic gradient lattice (SGL) joint process estimator filter structure for trained adaptive equalization was investigated in [5]. The SGL was combined with CMA and investigated for interference rejection in [8]. Lattice CMA was investigated for blind equalization in [9]. Here we introduce a lattice implementation of MMA which is very similar to the lattice CMA investigated in [9]. The structure of the lattice
CMA and MMA equalizers is shown in Figure 3. The lattice order recursion update equations are given by
\[ f_{e}(n) = f_{e}(n) - K_{e}(n) b_{e}(n-1), \] (14)
and
\[ b_{e}(n) = b_{e}(n-1) - K_{e}(n) f_{e}(n), \] (15)
1 \leq t \leq L - 1, where \( K_{e}(n) \) is the \( e \)-stage PARCOR coefficient, and \( f_{e}(n) \) and \( b_{e}(n) \) are the \( e \)-order forward and backward prediction-errors, respectively, all at time \( n \). Note that \( f_{e}(n) = b_{e}(n) = u(n) \). The \( e \)-stage PARCOR coefficient is updated according to
\[ K_{e}(n+1) = K_{e}(n) + \hat{P}_{e}^{-1}(n)[\hat{f}_{e}^{p}(n-1)f_{e}(n) + f_{e}(n)b_{e}^{n}(n)], \] (16)
1 \leq t \leq L - 1, where
\[ \hat{P}_{e}(n) \triangleq (1 - \alpha_{e})\hat{P}_{e}(n-1) + \left[ f_{e}(n)^{T} + b_{e}(n-1) \right] \] (17)
is an estimate of the total prediction-error power entering the \( e \)-stage, and \( 0 < \alpha_{e} < 1 \) is a normalized step size parameter. \( \hat{P}_{e}(n) \) serves as a power-normalized step-size in the update (16). The output of the lattice CMA and MMA (LCMA and LMMA, respectively) equalizers, is given by the inner product
\[ \hat{a}(n) = g^{T}(n)b(n) = \sum_{t=0}^{L-1} g_{t}(n)b_{t}(n) \] (18)
where \( g(n) \triangleq [g_{0}(n) \ldots g_{L-1}(n)]^{T} \) is the vector of adaptive lattice tap-weights at iteration \( n \). The LCMA and LMMA stochastic gradient updates for the \( e \)-variable lattice tap-weight \( g_{t}(n) \) are
\[ g_{t}(n+1) = g_{t}(n) - \hat{P}_{e}^{-1}(n)g_{t}(n)b_{e}(n), \] (19)
and
\[ g_{t}(n+1) = g_{t}(n) - \hat{P}_{e}^{-1}(n)e_{CMA}(n)b_{t}(n), \] (20)
respectively, where
\[ \hat{P}_{e}(n) \triangleq (1 - \alpha_{e})\hat{P}_{e}(n-1) + b_{e}(n) \] (21)
is an estimate of the \( e \)-order backward prediction-error power, \( 0 < \alpha_{e} < 1 \) is the normalized step-size parameter (same as in (17)), and \( e_{CMA}(n) \) and \( e_{CMA}(n) \) are defined in (9) and (12), respectively. \( \hat{P}_{e}^{-1}(n) \) serves as a power-normalized step-size in the updates (19) and (20). The power estimates defined in (17) and (21) are biased and over-estimate the true power quantity by a factor of \( \alpha_{e}^{2} \). Reducing \( \alpha_{e} \) decreases the step-size and the misadjustment of the lattice algorithms. Increasing \( \alpha_{e} \) allows the algorithms to converge more quickly at the expense of increasing the misadjustment. It is also possible to replace \( \mu \) in the transversal updates defined in (2) and (11) with the power-normalized step-size \( \hat{P}_{e}(n) \)
\[ \hat{P}_{e}(n) \triangleq (1 - \alpha_{e})\hat{P}_{e}(n-1) + u^{T}(n)u(n) \] (22)
is a running estimate of the total power in the transversal equalizer, and \( 0 < \alpha_{e} < 1 \) is the normalized step-size parameter. Varying \( \alpha_{e} \) has the same general effects on the convergence time and asymptotic MSE of the transversal implementations as varying \( \alpha_{e} \) does on the performance of the lattice implementations.

4 Misadjustment in MMA and CMA

A widely used figure-of-merit for the asymptotic performance of an adaptive algorithm is the misadjustment [6]
\[ \mathcal{M} \triangleq \left( \frac{e^{2}_{\min} - e^{2}_{\min}}{e^{2}_{\min}} \right) \] (23)
where \( e^{2}_{\min} \) is the asymptotic mean-square error (MSE) achieved by the algorithm and \( e^{2}_{\min} \) is the MSE achieved by the optimum Wiener filter with fixed tap-weights. \( \mathcal{M} \) is non-zero whenever the tap-weights continue to jitter about their optimum settings after the update algorithm has converged to a stationary point. Consider an ideal situation in which the equalizer perfectly compensates for the communications channel and there is no thermal noise (i.e., \( \hat{a}(n) = a(n)\sigma(n) = a(n-d) \) and \( \nu(n) = 0 \) in Figure 1). Then, if the transmitted signal is PSK \( \{ \ldots, R_{2} = r_{0} = \sigma(n) \} \), \( e_{CMA}(n) = 0 \) for all values of \( \hat{a}(n) \) and the CMA update equation does not move the adaptive weights; they stay fixed at their stationary points. If, on the other hand, the transmitted signal is QAM, \( e_{CMA}(n) \neq 0 \) for all values of \( \hat{a}(n) \) and the CMA update equation does move the adaptive weights. For QAM, because of this mismatch between the constellation and the cost function, the CMA update equation causes the adaptive weights to jitter about their stationary points even if perfect equalization is achieved in a noise-free setting. Therefore, under any conditions, not only in this ideal case, one would expect the misadjustment of CMA to be greater for QAM signals than for PSK signals, all else being equal. In [10], it is shown that the portion of the variance of a converged CMA tap-weight that is caused by the CMA cost function - QAM constellation mismatch is proportional to two terms. These terms are
\[ \gamma_{1} \triangleq E\left[|\sigma_{1}|^{2}(\sigma_{1} |R_{2}|)^{2}\right] \] (24)
and
\[ \gamma_{2} \triangleq E\left[|\sigma_{1}|^{2}|(\sigma_{1} |R_{2}|)^{2}\right]. \] (25)
where the expected values in (24) and (25) are evaluated over the constellation in question under the assumption that all symbols are equally likely. As mentioned in Section 1, for PSK \( R_{2} = |\sigma_{1}|^{2} \). Thus both (24) and (25) are zero for PSK. Indeed, there is no mismatch between the CMA cost function and a PSK constellation. Figure 4 plots (24) and (25) for the various QAM constellations referred to in Section 2. Depending on the constellation, either term can be slightly larger than the other. In [10], a similar result is derived for MMA and the terms which correspond to (24) and (25) are
\[ x_{1} \triangleq E\left[|\sigma_{1}|^{2}(\prod_{n}^{L}(|\sigma_{1} |\sigma_{1}| - \tau_{1}|)^{2}\sum_{n=2}^{L}(|\sigma_{1} |\sigma_{1}| - \tau_{1}|)^{2}\right] \] (26)
and
\[ x_{2} \triangleq E\left[|\sigma_{1}|^{2}(\prod_{n}^{L}(|\sigma_{1} |\sigma_{1}| - \tau_{1}|)^{2}\sum_{n=2}^{L}(|\sigma_{1} |\sigma_{1}| - \tau_{1}|)^{2}\right]. \] (27)
It is easy to show that (26) and (27) are both zero [10]. There is no mismatch between the MMA cost function and the QAM constellation. As a result, the variance of a converged MMA tap-weight for perfect equalization in the absence of thermal noise is zero. Under any conditions, one would expect the misadjustment of MMA to be less than that of CMA for QAM.

5 Simulation Experiments

Figure 5 plots four CMA and four MMA learning curves for the 8-QAM constellation referred to in Section 2. Each algorithm was implemented with an \( L = 11 \) tap transversal filter. The simulations that produced these learning curves used the channel model in (4) with \( W = 2.9, 3.1, 3.3 \) and 3.5, and a thermal noise level of \( \sigma_n^2 = 0.002 \). The four horizontal dashed lines in Figure 5 indicate \( \varepsilon_\text{min}^2 \) for an 11-tap equalizer with these four values of \( W \) and this thermal noise level. The adaptive filters were initialized with all taps set to zero except the center tap, \( \ell = 5 \), which was set to 1. The fixed step-size used in each algorithm was \( \mu = 2 \times 10^{-4} \). Each plotted learning curve represents an average of 10 runs. In Figure 5, for either algorithm, the four learning curves correspond to \( W = 2.9, 3.1, 3.3 \) and 3.5, respectively, moving from left to right. Several observations can be made from these results. Note that, although \( \varepsilon_\text{min}^2 \) ranges over a factor of 0.0113/0.0030 = 3.77 for the four values of \( W \), the asymptotic MSE achieved by CMA, \( \varepsilon_\infty^2(\text{CMA}) \), ranges only over a factor of 0.0518/0.0420 = 1.23. This indicates that the CMA cost function - QAM constellation mismatch, rather than \( \varepsilon_\text{min}^2 \), is the dominant factor in determining \( \varepsilon_\infty^2(\text{CMA}) \). In contrast, the asymptotic MSE achieved by MMA, \( \varepsilon_\infty^2(\text{MMA}) \), ranges over a factor of 0.0133/0.0035 = 3.80, roughly the same as \( \varepsilon_\text{min}^2 \). The achieved value of \( \varepsilon_\infty^2(\text{MMA}) \) more closely tracks the conditions imposed by the channel than does \( \varepsilon_\infty^2(\text{CMA}) \). Accordingly, the misadjustment \( \mathcal{A}(\text{MMA}) \) achieved by MMA is roughly constant at a level of 0.18 for the four values of \( W \). In contrast, \( \mathcal{A}(\text{CMA}) \) actually decreases for 8-QAM as \( W \) rises since the value of \( \varepsilon_\text{min}^2 \) rises much more quickly than does \( \varepsilon_\infty^2(\text{CMA}) \). Nonetheless, \( \mathcal{A}(\text{CMA}) > \mathcal{A}(\text{MMA}) \) by factors ranging from 19.8 to 76, i.e., roughly 13 to 19 dB. Finally, note that, within about 5\%, \( \tau_\infty(\text{MMA}) = \tau_\infty(\text{CMA}) \) for all four values of \( W \), where \( \tau_\infty \) is the number of iterations required to reach \( \varepsilon_\infty^2 \).

Figure 6 plots four transversal CMA and four transversal MMA learning curves for the 16-QAM constellation referred to in Section 2. The simulations that produced these learning curves were conducted under the same conditions as those discussed in the previous paragraph, but the step-size used in each algorithm was \( \mu = 5 \times 10^{-5} \) for 16-QAM. Again, several observations can be made from the results. Although \( \varepsilon_\text{min}^2 \) ranges over a factor of 0.0125/0.0030 = 4.17 for the four values of \( W \), \( \varepsilon_\infty^2(\text{CMA}) \) ranges only over a factor of 0.1047/0.093 = 1.12. This indicates that, as for 8-QAM, the CMA cost function - QAM constellation mismatch is the dominant factor in determining \( \varepsilon_\infty^2(\text{CMA}) \) for 16-QAM. \( \varepsilon_\infty^2(\text{MMA}) \) ranges over a factor of 0.0167/0.0042 = 3.98, roughly the same as \( \varepsilon_\text{min}^2 \). \( \mathcal{A}(\text{MMA}) \) varies slightly over a range from 0.34 to 0.42. As it did for 8-QAM, \( \mathcal{A}(\text{CMA}) \) decreases for 16-QAM as \( W \) rises since, again, the value of \( \varepsilon_\text{min}^2 \) rises much more quickly than does \( \varepsilon_\infty^2(\text{CMA}) \). Again, as for 8-QAM, \( \mathcal{A}(\text{CMA}) > \mathcal{A}(\text{MMA}) \) by factors ranging from roughly 13 to 19 dB for 16-QAM. However, the aspects of the 16-QAM results in Figure 6 that contrast greatly with the 8-QAM results in Figure 5 have to do with \( \tau_\infty(\text{MMA}) \). As evidenced by the rapid increase of \( \tau_\infty(\text{MMA}) \) with \( W \) in Figure 6, MMA with \( \rho = 3 \) signal point radii exhibits much greater sensitivity to the increasing eigenvalue disparity of \( \mathbf{R} \) than do either CMA, or MMA with \( \rho = 2 \) signal point radii. Indeed, for \( W = 3.5 \), \( \tau_\infty(\text{MMA}) = 52,000 \) whereas \( \tau_\infty(\text{CMA}) = 13,200 \). This issue will be addressed further below when the lattice implementation of MMA is applied to 16-QAM.

In order to compare the convergence performance of transversal and lattice implementations of CMA and MMA (TCMA, LCMA, TMMA and LMMA) independently of the values of the step-size parameters, several smoothed learning curves were produced for each algorithm using values of \( \alpha_L \) and \( \alpha_T \) (the transversal implementations discussed in this paragraph employed a normalized step-size per (22)) ranging from the largest that can be used for stability to values smaller than would normally be used in practice (since such small values serve only to lengthen the convergence time without significantly reducing the asymptotic MSE). Following the techniques given in [11], from these learning curves, the asymptotic MSE \( \varepsilon_\infty^2 \) and the number of iterations required to reach \( \varepsilon_\infty^2 \) were determined and plotted producing convergence performance curves. When comparing two adaptive equalization algorithms in this way, if one algorithm's performance curves always lie below and to the left of those of another, then the former algorithm clearly has superior convergence performance. Convergence performance comparisons were carried out via simulation using the channel model defined in (4) with \( W = 2.9, 3.1, 3.3 \), and 3.5. Figures 7 and 8 show convergence performance curves for 11-tap transversal and lattice implementations of CMA and MMA operating on 8-QAM and 16-QAM, respectively. All the equalizers were initialized with all taps set to zero except the center tap, \( \ell = 5 \), which was set to 1. Several observations can be made from these figures. Most notably, the \( \varepsilon_\infty^2 \) s exhibited by both TMMA and LMMA are considerably less than those exhibited by TCMA and LCMA for both modulations over the full dynamic range of eigenvalue disparities. In comparing the two implementations of MMA, LMMA exhibits a significant convergence performance advantage over TMMA for the larger eigenvalue disparities for both modulations, but especially for 16-QAM. As demonstrated above, TMMA's sensitivity to increasing eigenvalue disparity is much greater than that of.
TCMA for 16-QAM. This is evidenced here by the fact that
TMMA's convergence performance curve moves more rapidly
to the right (longer convergence times) through the four panels
of Figure 8 than does TCMA's performance curve. For both 8-
and 16-QAM, LMMA is a viable alternative to LCMA because
it offers significantly reduced misadjustment. LMMA's
convergence time does, however, exhibit more sensitivity to
increasing eigenvalue disparity for 16-QAM than for 8-QAM,
but the difference is not nearly as dramatic as it is for TMMA.

6 Conclusions
When CMA is applied to QAM, its misadjustment has
components that depend on the mismatch between Godard's
cost function and the constellation. The magnitudes of these
components increase monotonically with the number of radii in
the QAM constellation. The equivalent cost function -
constellation mismatch misadjustment components for MMA
are equal to zero. The misadjustment performance of MMA for
QAM is comparable to the misadjustment performance of CMA
for PSK [10]. Unfortunately, the practical limit of the number of
radii to which MMA can be applied appears to be two or
three for standard QAM constellations (8- and 16-QAM). This
limit is due to the large dynamic range of the maximum
magnitudes of the values taken on by eq. (8) for larger numbers
of radii. For two constellation radii, MMA provides a viable
alternative to CMA that might be compelling because of the
dramatic reduction in misadjustment (13-19 dB was
demonstrated here). For three constellation radii, however,
the convergence time of the transversal implementation of MMA
exhibits exaggerated sensitivity to increasing amplitude
distortion compared with CMA. Lattice implementations of
CMA and MMA exhibit reduced sensitivity to amplitude
distortion. The lattice is particularly useful in the application
of MMA to 16-QAM. An alternative approach to blind
equalization of 16-QAM (and higher order QAM constellations)
that also exhibits reduced misadjustment, without exhibiting exaggerated sensitivity to amplitude
distortion, is presented in [10].

7 References
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8 Figures

Figure 1 Setup.

(a) 8-QAM
(b) 16-QAM
(c) 32-QAM
(d) 64-QAM

Figure 2 MMA error function magnitudes vs. the magnitude
of the equalizer output for four standard QAM constellations.
Figure 3 Structure of the lattice CMA and MMA equalizers. The CMA and MMA error function components are given by eqs. (9) and (12), respectively.

Figure 4 Components of the variance of the CMA tap-weight update after convergence.

Figure 5 CMA and MMA learning curves for 8-QAM.

Figure 6 CMA and MMA learning curves for 16-QAM.

Figure 7 Performance curves for 8-QAM.

Figure 8 Performance curves for 16-QAM.