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Vector Quantization Techniques for Feedback Based MIMO Systems

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Current Research Areas

- Feedback in MIMO systems
- Multi-User MIMO systems and Radio Resource Management
- Training and Channel Estimation Techniques
  - Semi-Blind Techniques
- Receiver Design and Analysis
  - Joint Channel Estimation and Decoding
- Capacity of Multi-Cellular systems

Motivation

- Increased Capacity Compared to Systems without Feedback
- Enables Interference Minimization
- Better Radio Resource Management
- Reduces Power Consumption
- Stealth of User and Resource Position
- Data Security

Research Issues

- Quantization Criteria
- Codebook Design
- Performance Evaluation
- Complexity vs Performance Tradeoffs
- Dynamic Channels (Temporal Correlation)

Introduction

- Transmitter's knowledge of channel can be utilized to improve the link performance of MIMO channels (e.g., capacity or error probability).
- In many cases of study, the channel information at transmitter was assumed to be perfect, i.e., no quantization error.
- **Question:** How can we efficiently feed back the MIMO channel information?
- Quantization of Spatial Information
  - Random unit-norm complex vector (MISO, MRT for MIMO)
  - Random unitary matrix or orthonormal column vectors (MIMO)

§§ VQ-based Approach for MISO Systems

- Beamforming for a MISO channel (t transmit antenna)

$$y = h^\dagger ws + \eta, \;\; \|w\| = 1$$

where $h^\dagger$ is the channel vector and w is beamforming vector.

$$w_{\text{opt}} = h/\|h\|$$

maximizes the received SNR and also the mutual information.

- **Problem:** Want to quantize the random unit-norm vector $v = h/\|h\|$ and feed back to Tx.

- Statistics of random vector v: For channel h with iid $\mathcal{CN}(0,1)$ entries, v is uniformly distributed over the unit-norm sphere.

$$v \sim \mathcal{U}(\mathcal{S}_t) \text{ where } \mathcal{S}_t = \{x \in \mathbb{C}^t : \|x\| = 1\}.$$

Capacity Loss due to Quantization

- Assume Tx uses $\hat{v} = \mathcal{Q}(v)$ $(\|\hat{v}\| = 1)$ for beamforming. Rx signal

$$y = \alpha \langle v, \hat{v} \rangle s + \eta, \quad \text{where } \alpha = \|h\| \text{ and } v = h/\|h\|.$$

- **Capacity Loss** due to quantization of beamforming vector

$$\begin{aligned} C_L(h, \hat{v}) &= C(h, v) - C(h, \hat{v}) \\ &= -\log\left(1 - \frac{\alpha^2 P_T}{1 + \alpha^2 P_T}\left(1 - |\langle v, \hat{v} \rangle|^2\right)\right) \end{aligned}$$

  where $C(h, w)$ is the capacity of channel h when Tx uses w as beamforming vector.

- When $|\langle v, \hat{v} \rangle|$ is close to one (e.g., when the quantizer has a reasonably high resolution), or when $P_T \ll 1$,

$$C_L(h, \hat{v}) \simeq \frac{\alpha^2 P_T}{1 + \alpha^2 P_T}(1 - |\langle v, \hat{v} \rangle|^2) =: \tilde{C}_L(h, \hat{v}).$$

New Quantizer Design Criterion

- **Design Criterion:** Design a quantizer to minimize $\tilde{C}_L(h, \hat{v})$, i.e.,

$$\max_{\mathcal{Q}(\cdot)} E \left| \langle \tilde{\alpha} v, \mathcal{Q}(h) \rangle \right|^2 \qquad (1)$$

where $\tilde{\alpha} = \sqrt{\frac{\alpha^2 P_T}{1+\alpha^2 P_T}}$ and $\hat{v} = \mathcal{Q}(h)$ is the quantized beamforming vector ($\|\hat{v}\| = 1$).

- **Two Related Design Criterions:**
  - In High SNR: as $P_T \to \infty$, $\tilde{\alpha} \to 1$, hence, (1) reduces to

$$\max_{\mathcal{Q}(\cdot)} E \left| \langle v, \mathcal{Q}(v) \rangle \right|^2.$$

  - In Low SNR: when $P_T \ll 1$, $\tilde{\alpha} \simeq \sqrt{P_T} \alpha$, (1) becomes Narula's

$$\max_{\mathcal{Q}(\cdot)} E \left| \langle h, \mathcal{Q}(h) \rangle \right|^2.$$

VQ Design Algorithm: Modified Lloyd Algr.

- **Nearest Neighborhood Condition (NNC):** For given code vectors $\{\hat{v}_i; i = 1, \ldots, N\}$, the optimum partition cells satisfy

$$\mathcal{R}_i = \left\{h \in \mathbb{C}^t : |\langle \tilde{\alpha} v, \hat{v}_i \rangle| \geq |\langle \tilde{\alpha} v, \hat{v}_j \rangle|, \forall j \neq i \right\}, \text{ for } i = 1, \ldots, N$$

  where $\mathcal{R}_i$ is the partition cell (Voronio region) for $\hat{v}_i$.

- **Centroid Condition (CC):** For given partition $\{\mathcal{R}_i; i = 1, \ldots, N\}$, the optimum code vectors satisfy

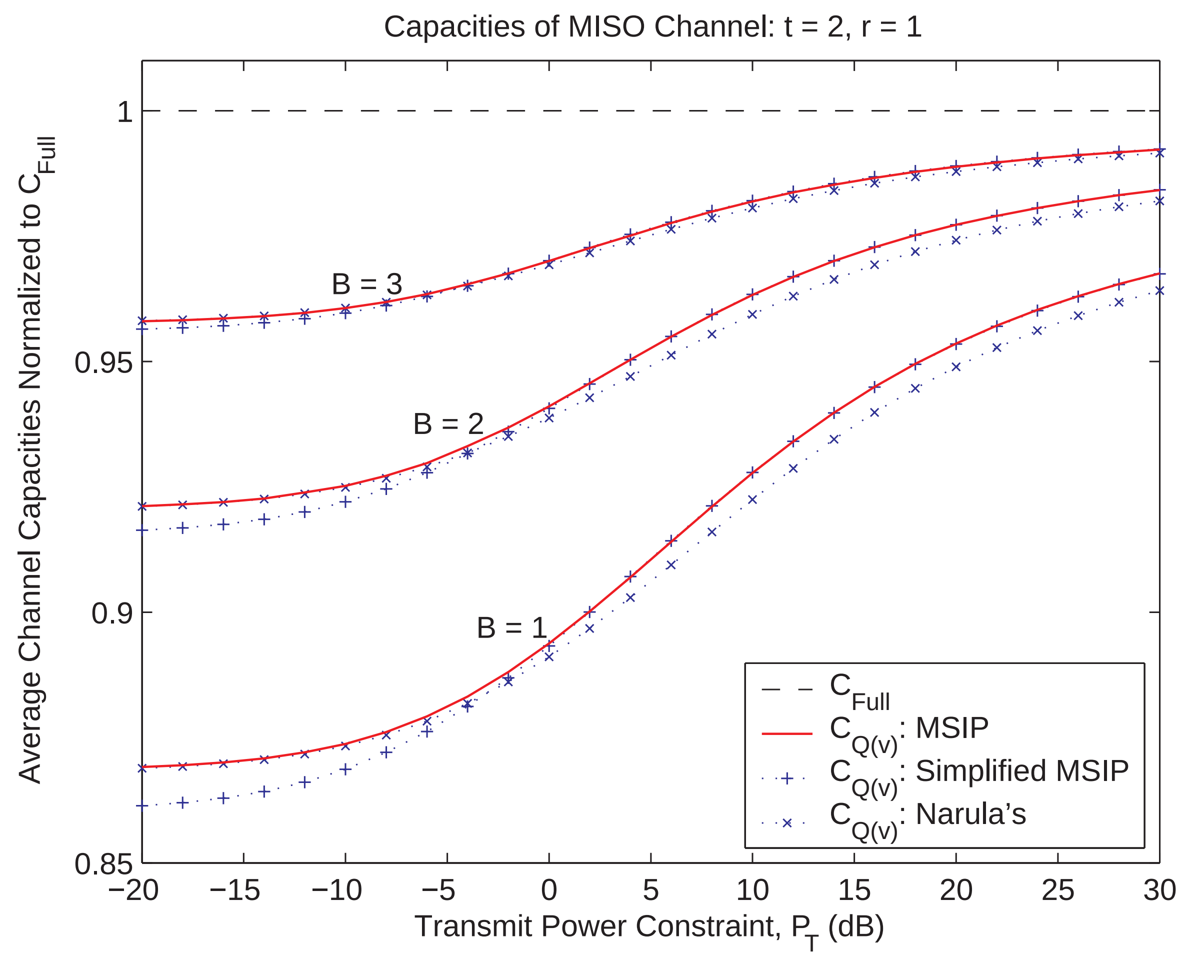
$$\begin{aligned} \hat{v}_i &= \arg \max_{\hat{v} \in \mathcal{R}_i,\, \|\hat{v}\|=1} E\left[|\langle \tilde{\alpha} v, \hat{v} \rangle|^2 \,\middle|\, h \in \mathcal{R}_i \right] \\ &= (\text{principal eigenvector}) \text{ of } E\left[\tilde{\alpha}^2 v v^\dagger | v \in \mathcal{R}_i\right], \\ &\quad \text{for } i = 1, \ldots, N. \end{aligned}$$

- Iterate the two conditions until the MSIP converges.

Codebook Design Results

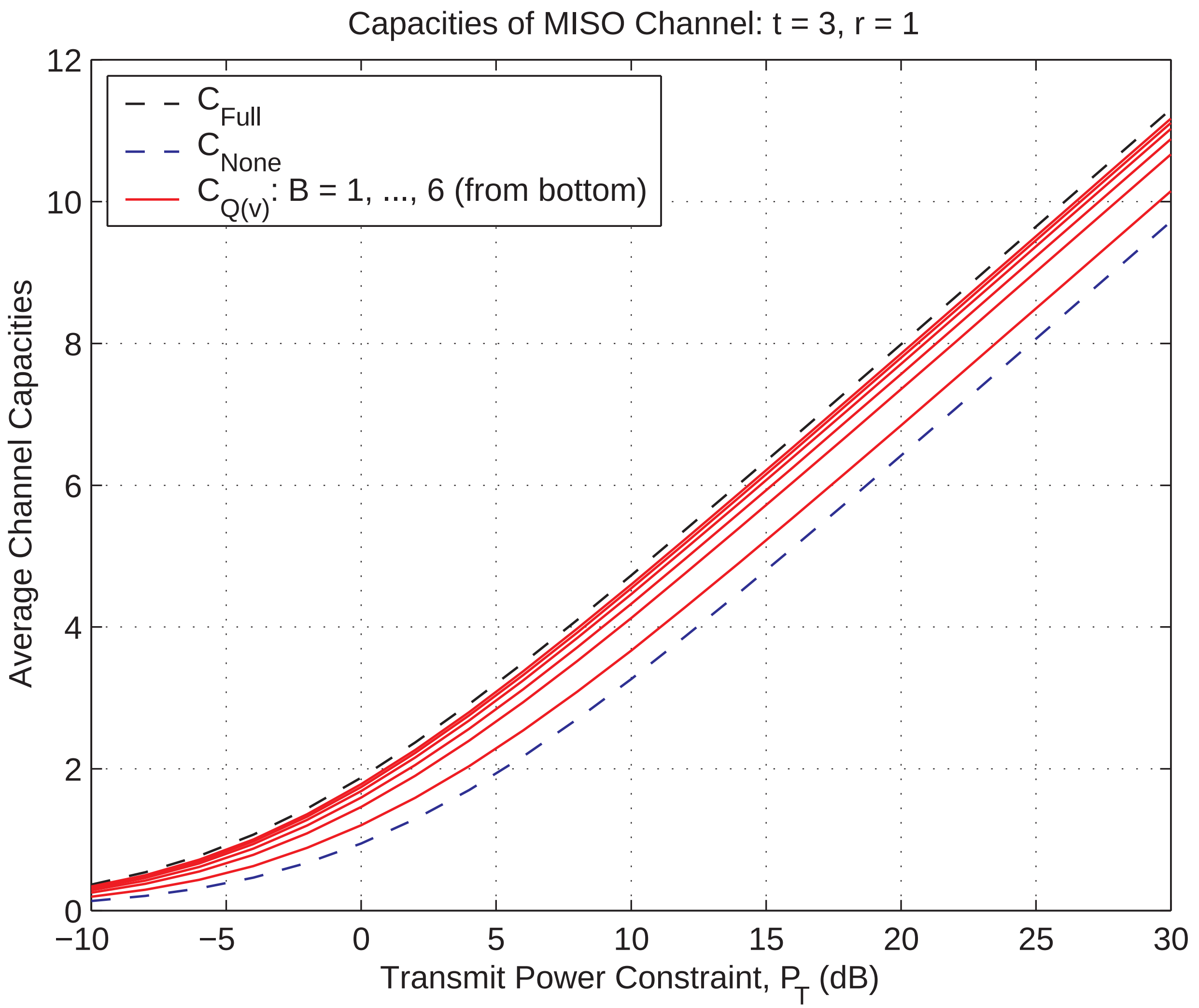
$$t = 2; N = 2^B, B = \{1, 2, 3\}$$

Antenna spacing of half-wavelength and uniform angular-spread in $[-30°, 30°]$



MISO Capacity with Quantized Feedback

$$t = 3, r = 1; N = 2^B, B = \{1, \ldots, 6\}$$



§§ Loss in Capacity with Quantized Feedback

- **Loss in Capacity**

$$\begin{aligned} C_L(h,\hat{v}) &= C(h,v) - C(h,\hat{v}) \\ &= \log(1+\sigma^2 P_T) - \log(1+\sigma^2 P_T |\langle v,\hat{v}\rangle|^2) \\ &= -\log\Big(1 - \frac{\sigma^2 P_T}{1+\sigma^2 P_T}(1-|\langle v,\hat{v}\rangle|^2)\Big) \end{aligned}$$

- **Average Loss in Capacity:** For a given $\mathcal{C} = \{\hat{v}_1, \ldots, \hat{v}_N\}, N = 2^B$,

$$\begin{aligned} C_L &= E_h[\, C_L(h,\hat{v}) \,] \\ &= \sum_{i=1}^{N} P(h \in \mathcal{R}_i)\, E_{h\in\mathcal{R}_i} C_L(h,\hat{v}_i) \\ &= -\sum_{i=1}^{N} P(h \in \mathcal{R}_i)\ \underbrace{E_{h\in\mathcal{R}_i} \log\Big(1 - \frac{\sigma^2 P_T}{1+\sigma^2 P_T}(1-|\langle v,\hat{v}_i\rangle|^2)\Big)}_{\star} \end{aligned}$$

Statistics of Unconstraint Inner Product

- Assume $h \sim \mathcal{CN}(0, I_t)$ and consider a random variable

$$\gamma_0 = |\langle v, v_0 \rangle|^2$$

for a fixed v_0 on the unit-sphere ($\|v_0\| = 1$).

- **Statistics of γ_0:**

$$\gamma_0 = \frac{|h^\dagger v_0|^2}{\|h\|^2} \overset{d}{=} \frac{\chi_2^2}{\chi_2^2 + \chi_{2t-2}^2} \sim \beta(1, t-1)$$

$$f_{\gamma_0}(x) = (t-1)(1-x)^{t-2},\ 0 < x < 1$$

- Similarly, $\xi_0 := 1 - \gamma_0 \sim \beta(t-1, 1)$

$$f_{\xi_0}(x) = (t-1)x^{t-2},\ 0 < x < 1$$

Quantization Cell Approximation

- Want to approximate the density function of $\gamma = |\langle v, \mathcal{Q}(v)\rangle|^2$ from the density of γ_0 (unconstraint).
- **Actual Quantization Cells:** For a given code book $\mathcal{C} = \{\hat{v}_1, \ldots, \hat{v}_N\}$, $N = 2^B$,

$$\mathcal{R}_i = \{v \in \mathcal{S}_t \ : \ |\langle v, \hat{v}_i\rangle|^2 > |\langle v, \hat{v}_j\rangle|^2, \ \forall j \neq i\}$$

- **Quantization Cell Approximation:**

$$\mathcal{R}_i \simeq \{v \in \mathcal{S}_t \ : \ \xi_i = 1 - \gamma_i = 1 - |\langle v, \hat{v}_i\rangle|^2 \leq \delta\}$$

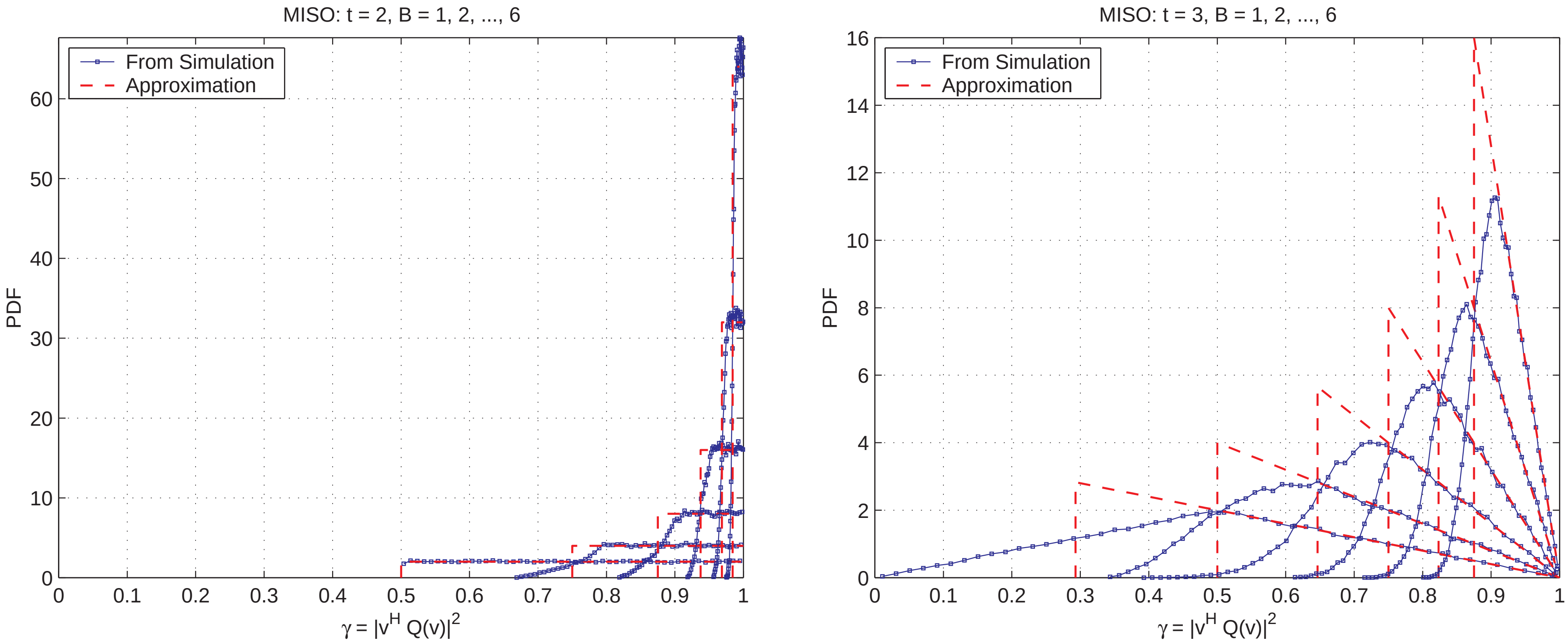
where $\delta = 2^{-\frac{B}{t-1}}$ from

$$P(v \in \mathcal{R}_i) \simeq P(\xi_i \leq \delta) = \int_0^\delta f_{\xi_0}(x)dx = \frac{1}{2^B}$$

Statistics of Inner Product with Quantization

✧ Approximated Density Function of $\gamma = |\langle v, \mathcal{Q}(v)\rangle|^2$:

$$f_\gamma(x) \simeq \begin{cases} 0, & 0 < x < 1-\delta \\ 2^B f_{\gamma_0}(x), & 1-\delta \le x < 1 \end{cases}$$



Average Loss in Capacity

- Average Loss in Capacity

$$C_L = -E_h \log\left(1 - \frac{\sigma^2 P_T}{1+\sigma^2 P_T}(1 - \underbrace{|\langle v, \mathcal{Q}(v)\rangle|^2}_{\gamma})\right)$$

We know $f_\gamma(x)$ and $\sigma^2 = \|h\|^2 \sim \Gamma(t,1)$

$$f_{\sigma^2}(x) = \frac{x^{t-1}e^{-x}}{\Gamma(t)}, \quad x > 0$$

- After some math,

$$\boxed{C_L = \frac{t-1}{\Gamma(t)} \sum_{k=1}^{\infty} \frac{\Gamma(k+t)}{k+t-1}\, 2^{-\frac{kB}{t-1}}\, {P_T}^k\, {}_2F_0(k+t, k;\,; -P_T)}$$

where ${}_2F_0(\cdot,\cdot;\,;\cdot)$ is the generalized hypergeometric function.

Approximations to Loss in Capacity

$$C_L(h,\hat{v}) = -\log\left(1 - \frac{\sigma^2 P_T}{1+\sigma^2 P_T}(1 - |\langle v,\hat{v}\rangle|^2)\right) \text{ (nats/sec/Hz)}$$

1. **High-Resolution Approximation:** when $|\langle v,\hat{v}\rangle| \simeq 1$,

$$C_L(h,\hat{v}) \simeq \frac{\sigma^2 P_T}{1+\sigma^2 P_T}(1 - |\langle v,\hat{v}\rangle|^2)$$

2. **High-SNR Approximation:** when $P_T \gg 1$,

$$C_L(h,\hat{v}) \simeq -\log|\langle v,\hat{v}\rangle|^2$$

3. **High-Resolution & High-SNR Approximation:**

$$C_L(h,\hat{v}) \simeq 1 - |\langle v,\hat{v}\rangle|^2$$

Approximations to Average Loss in Capacity

1. **High-Resolution Approximation:** when $|\langle v, \hat{v} \rangle| \simeq 1$,

$$C_L \simeq \frac{1}{\Gamma(t)} \left[\frac{(-1)^{t+1}}{{P_T}^t} e^{1/P_T} \mathrm{Ei}\left(-\frac{1}{P_T}\right) + \sum_{k=1}^{t} (k-1)! \left(-\frac{1}{P_T}\right)^{t-k} \right] \cdot \left(\frac{t-1}{t}\right) 2^{-\frac{B}{t-1}}$$

2. **High-SNR Approximation:** when $P_T \gg 1$,

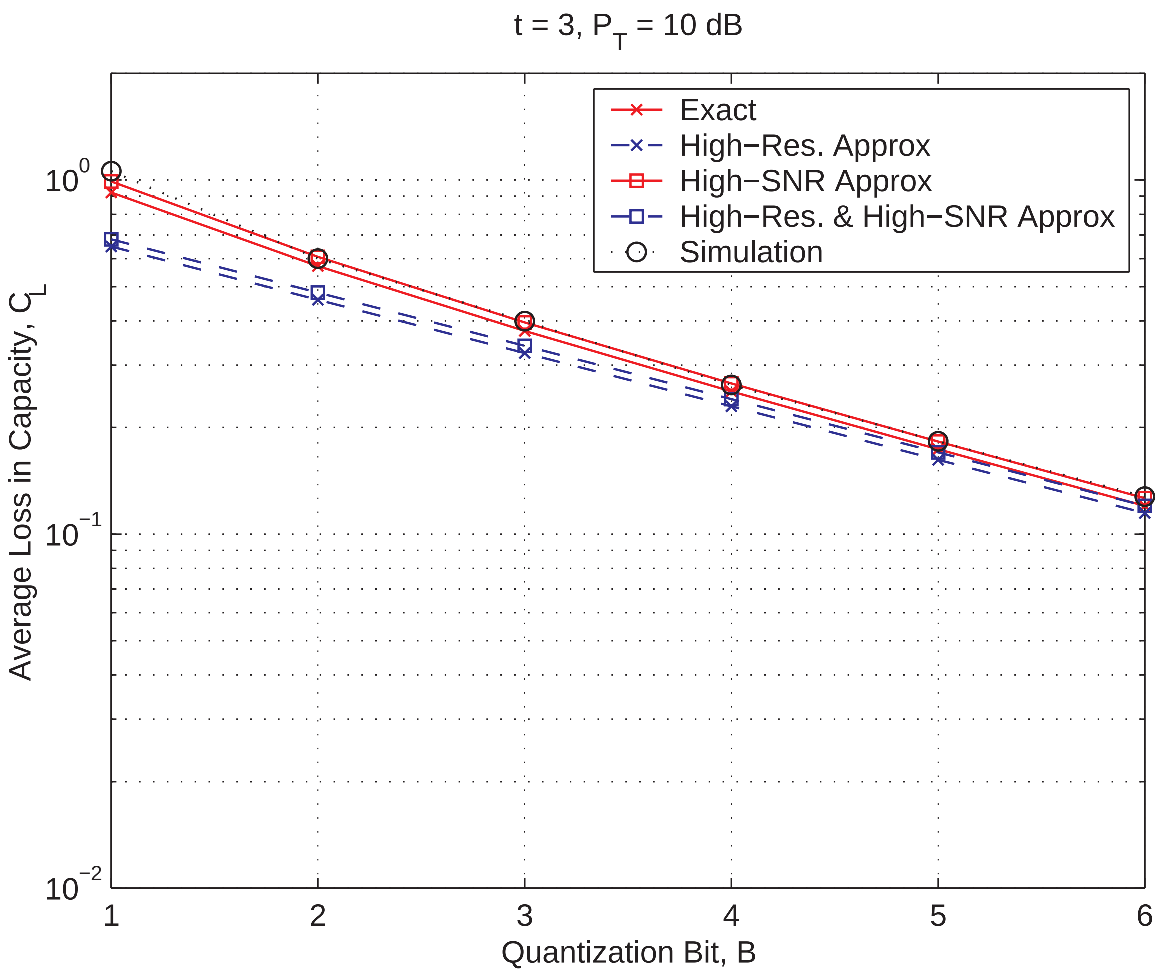
$$C_L \simeq 2^B \left[(1 - 2^{-B}) \log(1 - 2^{-B/(t-1)}) + \sum_{k=1}^{t-1} \frac{2^{-kB/(t-1)}}{k} \right]$$

3. **High-Resolution & High-SNR Approximation:**

$$\boxed{C_L \simeq \left(\frac{t-1}{t}\right) 2^{-\frac{B}{t-1}}}$$

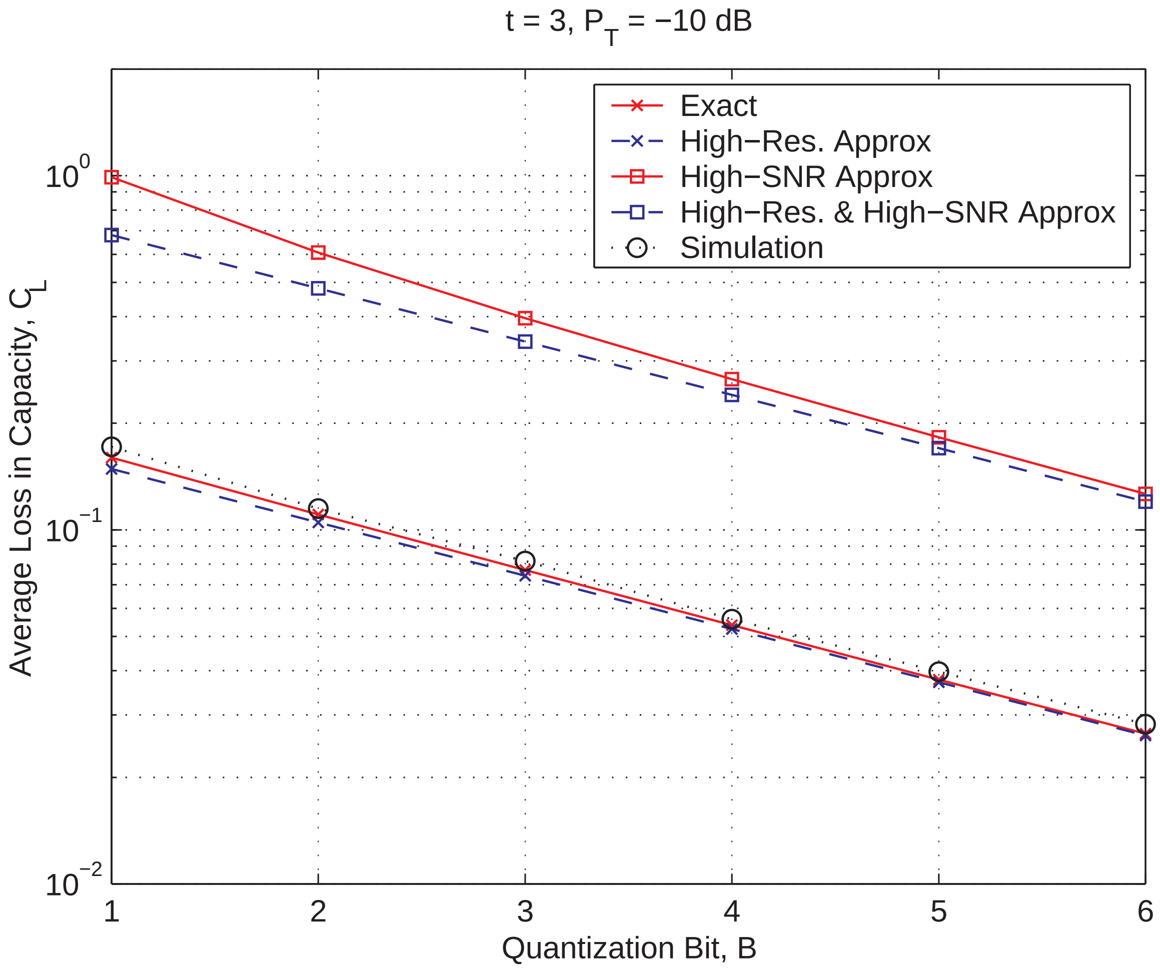
Average Loss in Capacity: Numerical Results

$t = 3; P_T = 10$ dB; $B = 1, 2, \ldots, 6$



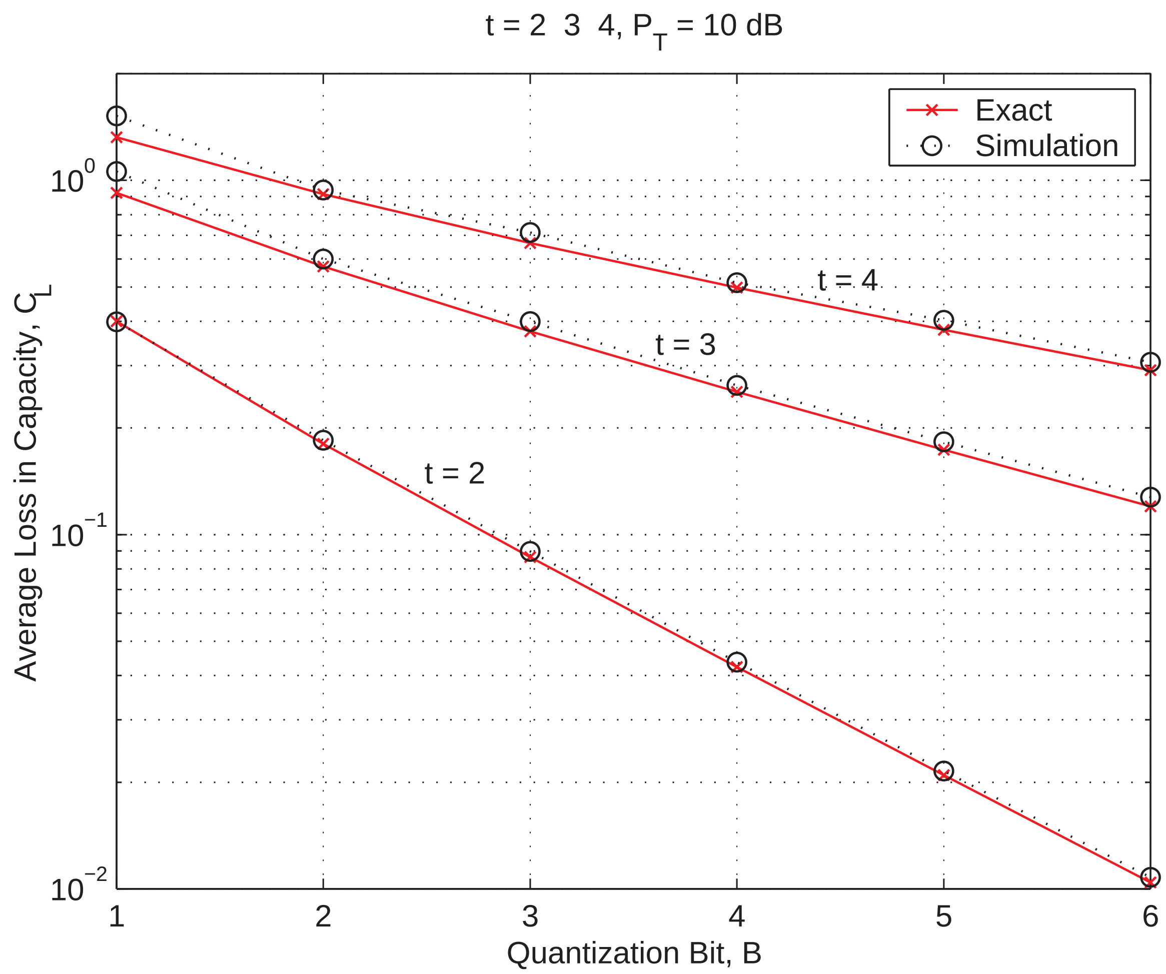
Average Loss in Capacity: Numerical Results

$t = 3; P_T = -10$ dB; $B = 1, 2, \ldots, 6$



Average Loss in Capacity: Numerical Results

$t = 2, 3, 4; P_T = 10$ dB; $B = 1, 2, \ldots, 6$



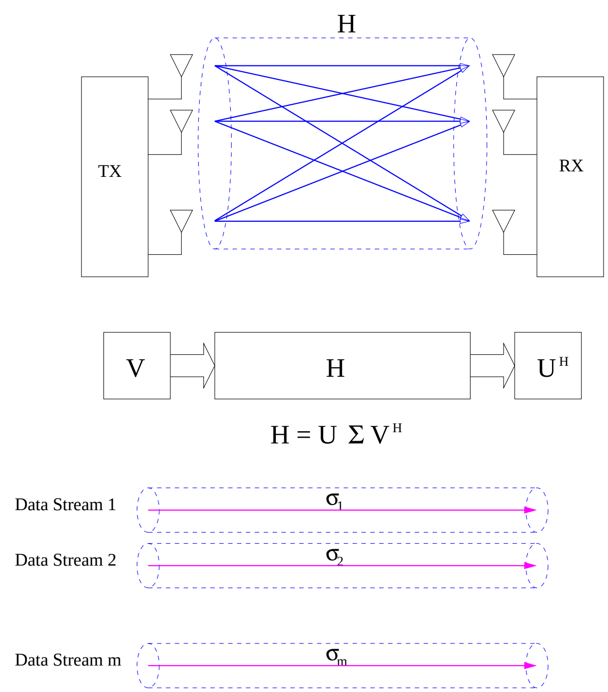
§§ Channel Feedback for MIMO Systems

- **Channel Model:** MIMO with t transmit and r receive antennas

$$y = Hx + \eta$$

$H \in \mathbb{C}^{r \times t}$ is channel gain matrix, $x \in \mathbb{C}^t$ transmit signal, $y \in \mathbb{C}^r$ receive signal, $\eta \in \mathbb{C}^r$ AWGN with $E[\eta\eta^\dagger] = I_r$, $E[x^\dagger x] \leq P_T$

- When the transmitter has perfect channel information



Channel Information to Feedback

- Channel capacity with Tx's perfect channel knowledge

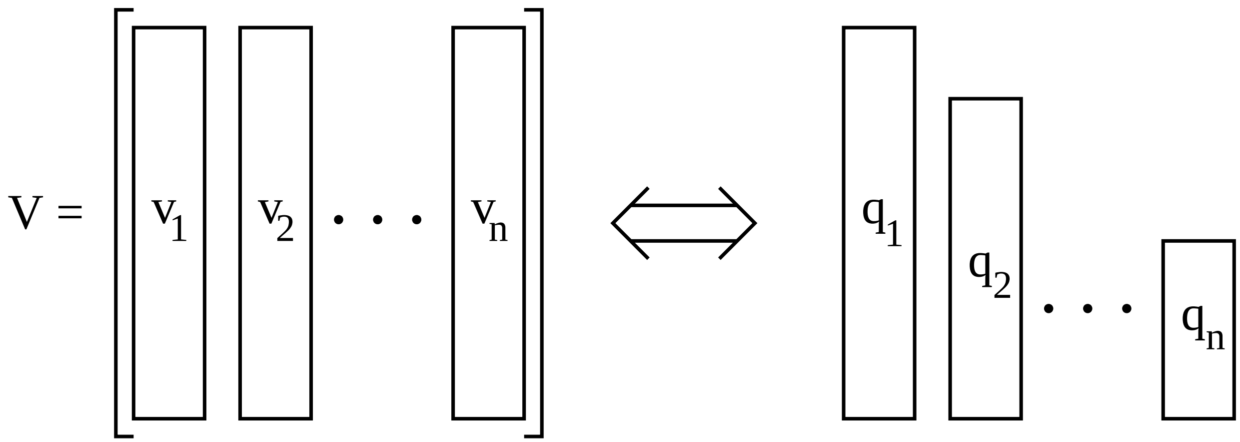
$$C_{HH}(P_T; H) = \max_{P_1+\ldots+P_m \le P_T} \sum_{i=1}^{m} \log(1 + P_i \lambda_i)$$

The optimum power allocation $P_1, P_2, \ldots, P_m (\sum_i P_i = P_T)$ is obtained by water-filling.

- **Problem:** Want to quantize and feed back $V = [v_1, \ldots, v_n]$ (Spatial Info.) and $\gamma = \frac{1}{P_T}[P_1, \ldots, P_n]$ (Power Alloc. for equivalent channel $H\hat{V}$), where $1 \le n \le \mathrm{rank}(H)$ for generality.

- **Geometric Structure in Spatial Information V:**
  - Each column is norm-one: $\|v_i\| = 1, \forall i$.
  - Columns are mutually orthogonal: $\langle v_i, v_j \rangle = v_i^\dagger v_j = 0$, if $i \neq j$.

Vector Parameterization



- **How?** Consider a unitary matrix G_1 whose first column is v_1 and the remainder columns are arbitrarily chosen to satisfy $G_1^\dagger G_1 = I$. Then, $G_1^\dagger V$ has the form of

$$G_1^\dagger V = \begin{bmatrix} 1 & 0 \\ 0 & V^{(2)} \end{bmatrix}$$

where $V^{(2)}$ is a $(t-1) \times (n-1)$ orthonormal column matrix.

The Quantization Method

- **Vector Parameterization:** An orthonormal column matrix $V \in \mathbb{C}^{t \times n}$ can be uniquely represented by a set of unit-norm vectors with different dimensions, $q_1 \in \mathcal{S}_t$, $q_2 \in \mathcal{S}_{t-1}$, . . ., $q_n \in \mathcal{S}_{t-n+1}$.
- **Statistical Property:** For random channel H with *i.i.d.* $\mathcal{CN}(0, 1)$ entries, $q_i \sim \mathcal{U}(\mathcal{S}_{t-i+1})$, for $i = 1, \ldots, n$, and they are statistically independent.
- **Quantization:** For $i = 1, \ldots, n$, unit-norm vector q_i is quantized using a codebook $\mathcal{C}_i$ that is designed for random unit-norm vector in $\mathbb{C}^{t-i+1}$ with the MSIP criterion.

Two Encoding Schemes

- **Encoding Scheme A (simple):** Encode sequentially $k = 1, \ldots, n$ as follows

$$\hat{q}_k = \arg\max_{\hat{q} \in \mathcal{C}_k} |\langle q_k, \hat{q} \rangle|.$$

- **Encoding Scheme B (better):** Noticing quantization errors are accumulating at each k, encode sequentially $k = 1, \ldots, n$

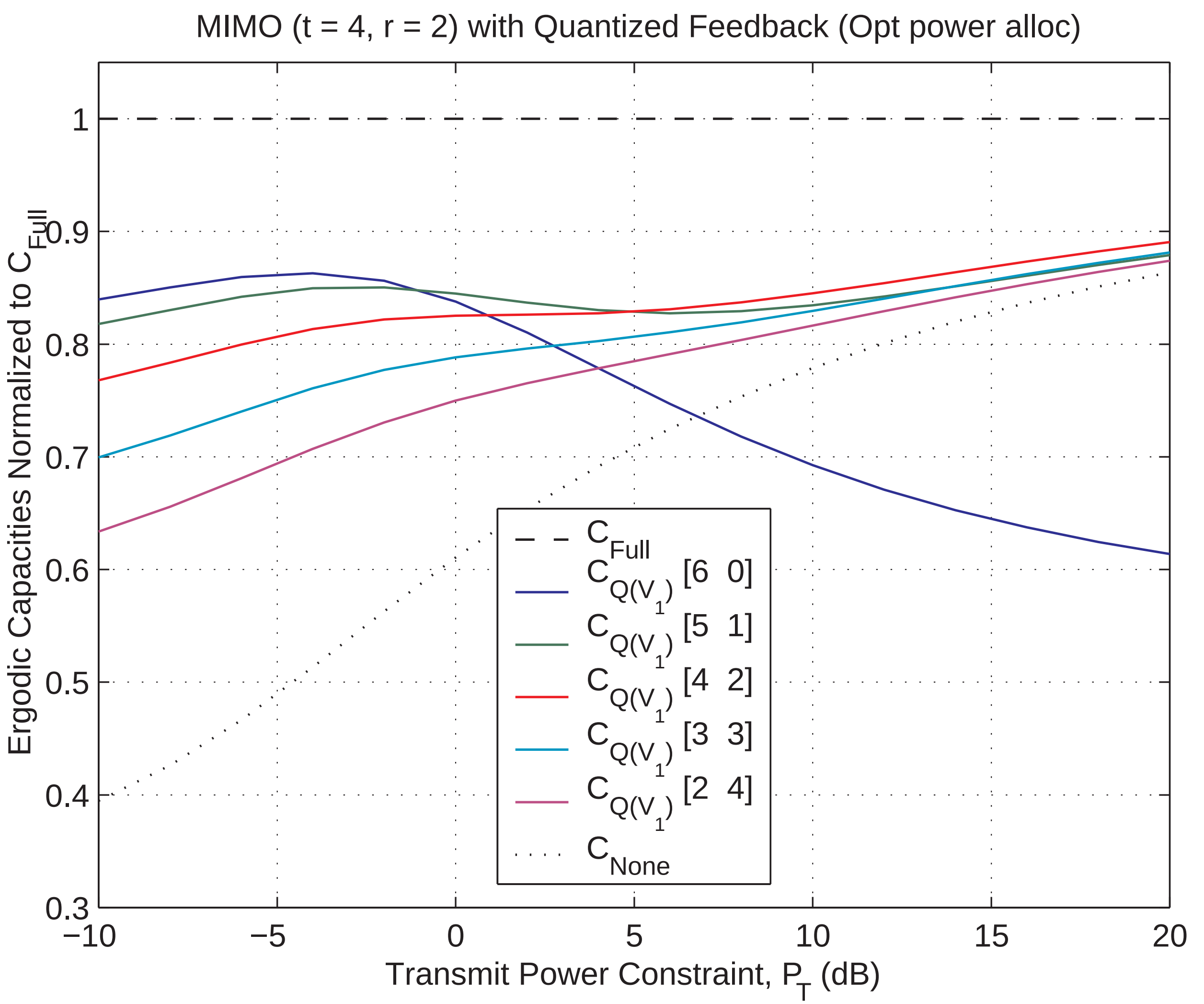
$$\begin{aligned} \hat{q}_k &= \arg\max_{\hat{q} \in \mathcal{C}_k} |\langle v_k, \hat{v} \rangle| \\ &= \arg\max_{\hat{q} \in \mathcal{C}_k} |\langle \tilde{q}_k, \hat{q} \rangle|. \end{aligned}$$

where $\hat{v} = \hat{G}_1 \cdots \hat{G}_{k-1} [0 \;\; \hat{q}]^T$, and $\tilde{q}_k$ is related with v_k as follows

$$\hat{G}_{k-1}^{\dagger} \cdots \hat{G}_1^{\dagger} v_k = \begin{bmatrix} a \\ \tilde{q}_k \end{bmatrix}, \; a \in \mathbb{C}^{k-1}.$$

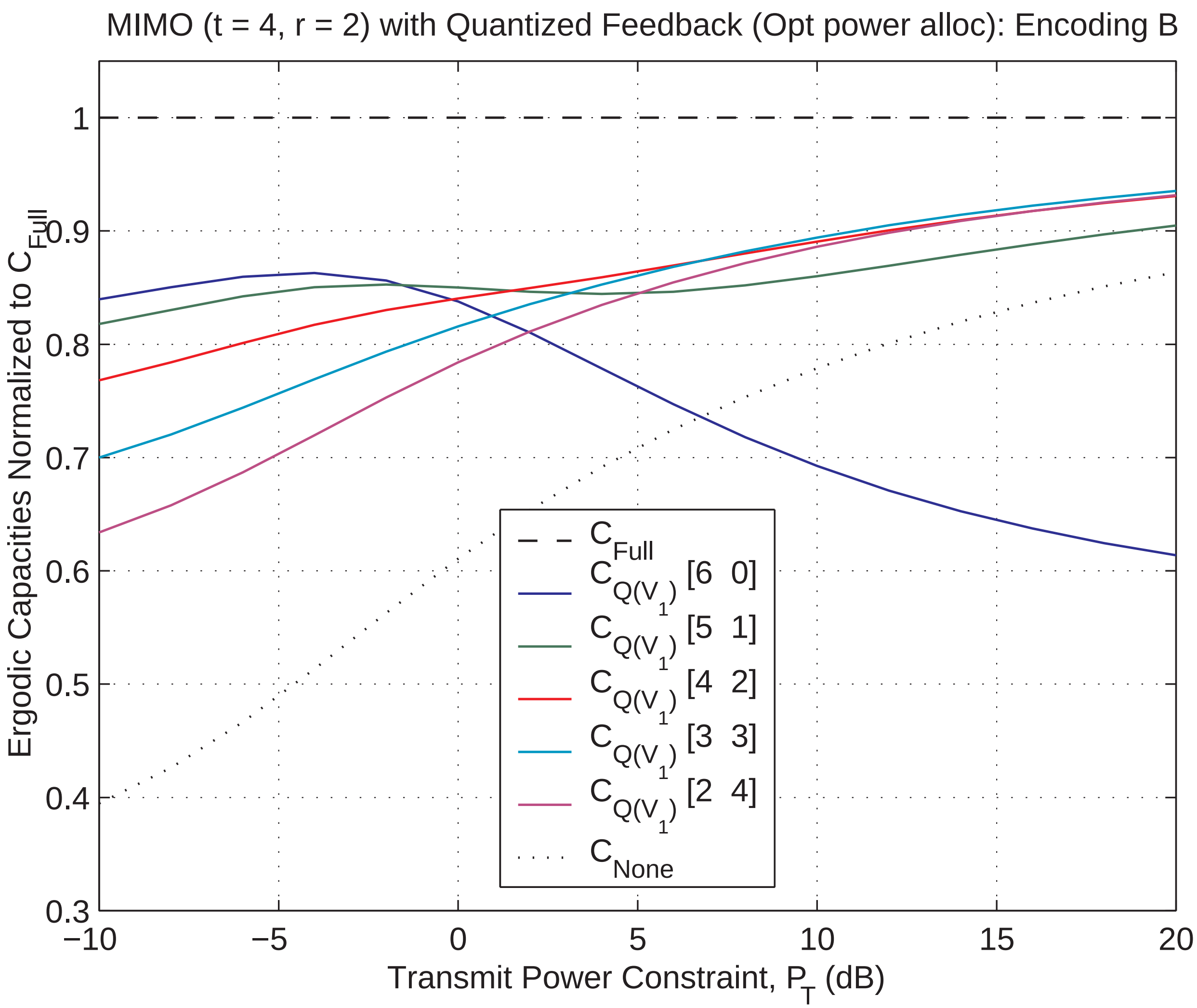
MIMO Capacity with Quantized Feedback

$t = 4, r = 2$; EncA; $B = 6$ bits, Bit Allocation: 6/0, 5/1, 4/2, 3/3, 2/4

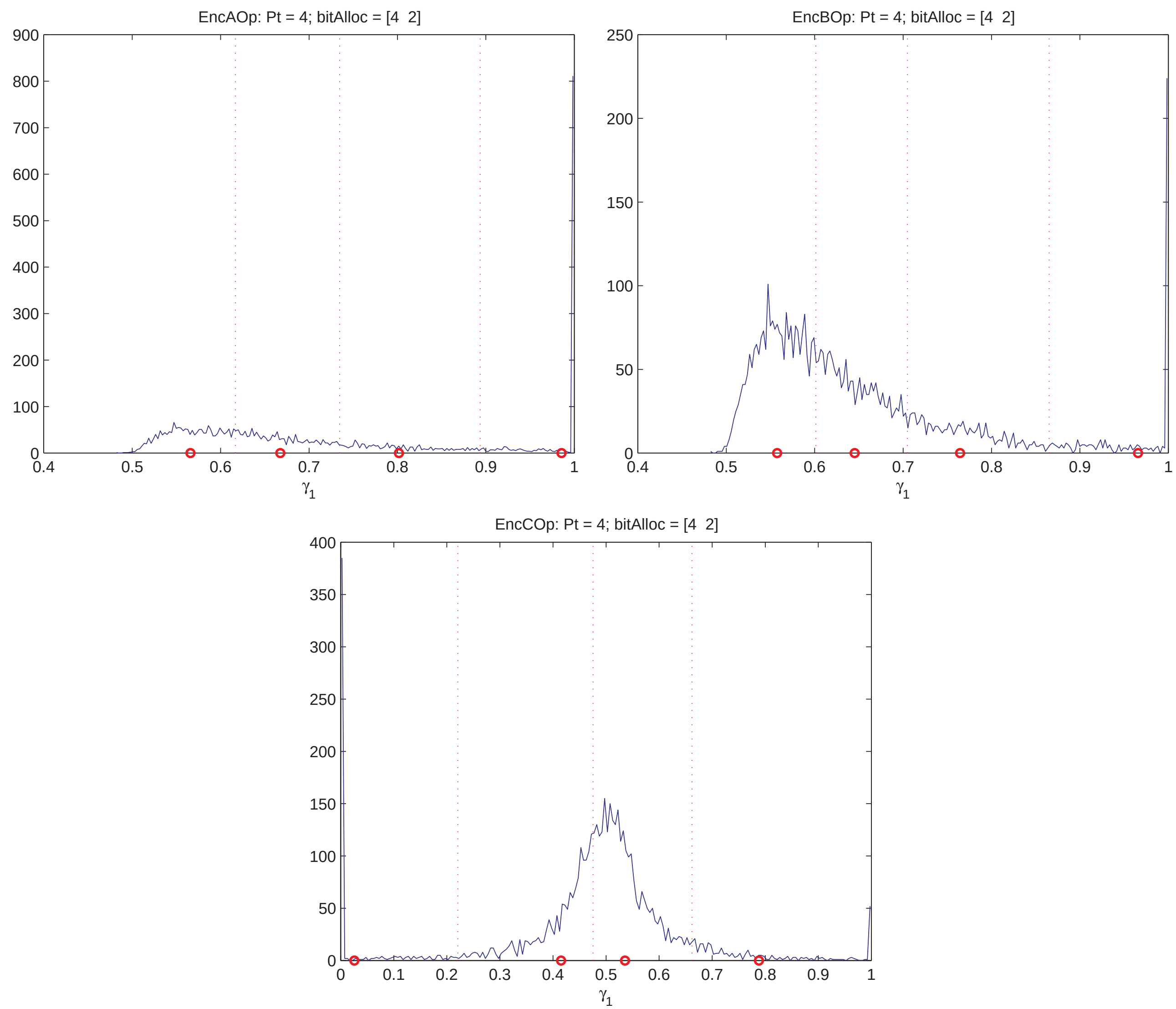


MIMO Capacity with Quantized Feedback

$t = 4, r = 2$; EncB; $B = 6$ bits, Bit Allocation: 6/0, 5/1, 4/2, 3/3, 2/4

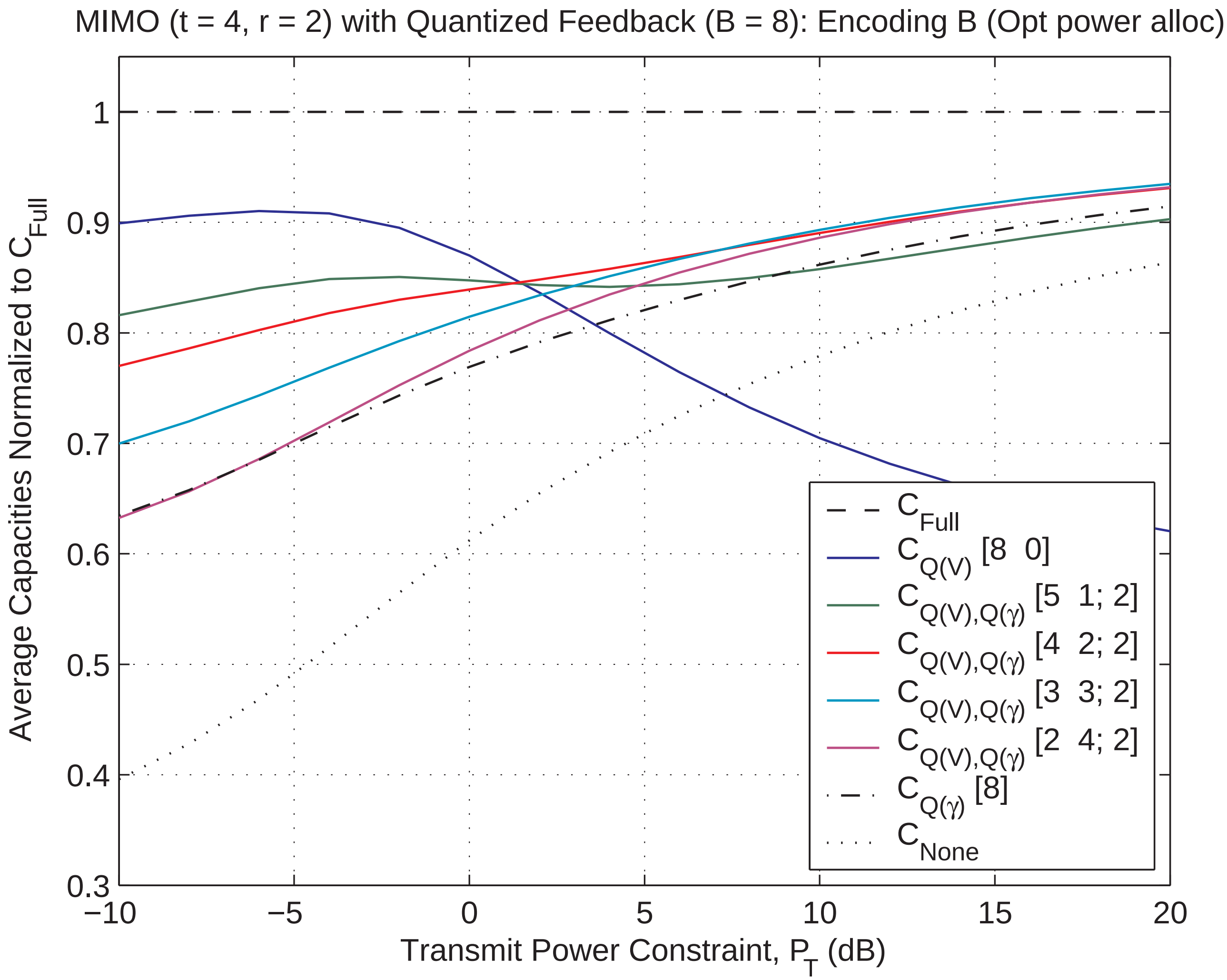


Quantization of Power Allocation Info: MSE



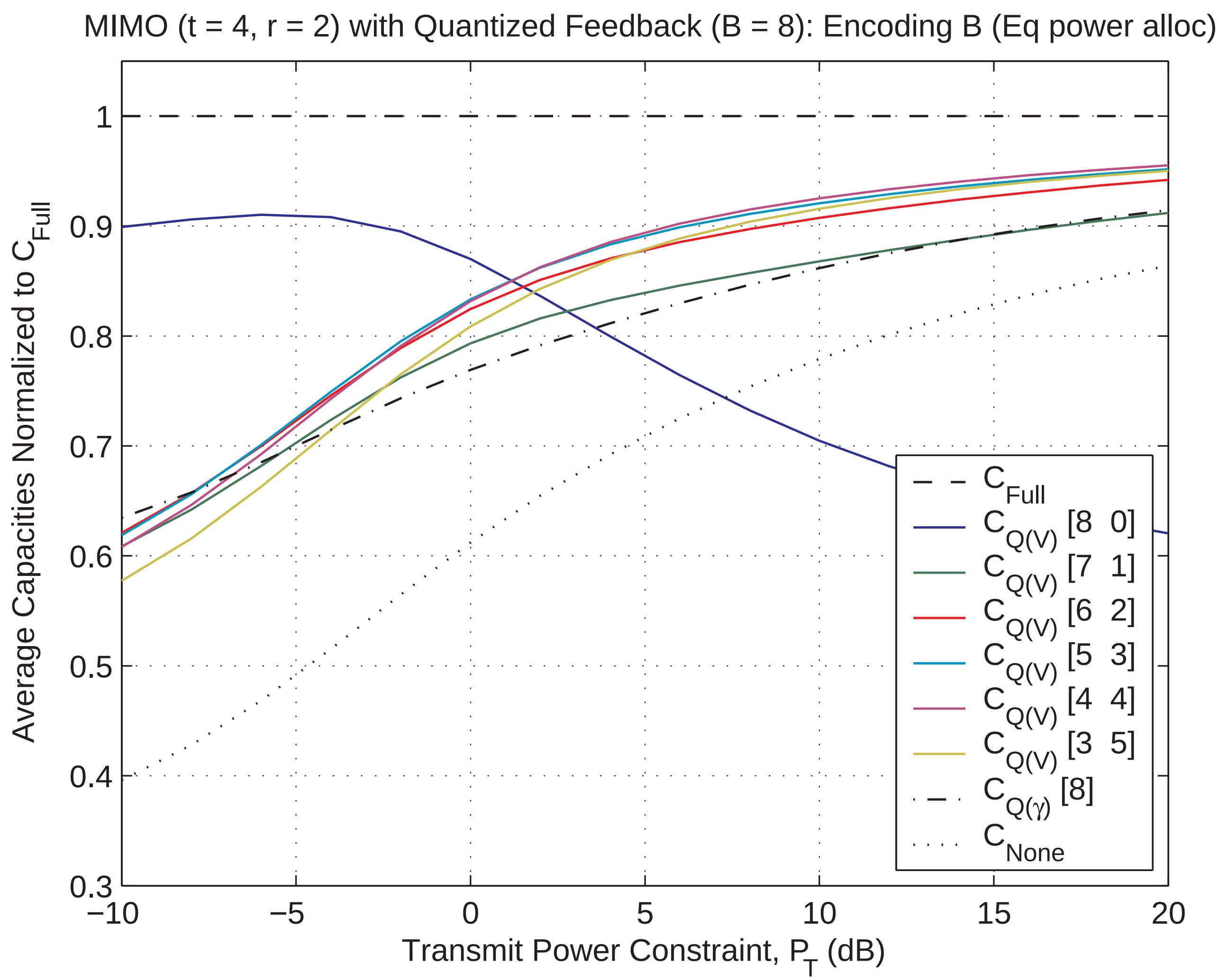
MIMO Capacity with Quantized Feedback

$t = 4, r = 2$; EncB; $B = 8$ bits; Quantized Optimum Power Allocation



MIMO Capacity with Quantized Feedback

$t = 4, r = 2$; EncB; $B = 8$ bits; Equal Power Allocation



Research Issues

- Quantization for MIMO systems
  - Optimal Joint Quantization of Beamforming Vectors
  - Quantization of Beamforming Vector + Power Allocation
  - Quantization Exploring Temporal Correlation
  - Complexity vs Performance Tradeoffs
  - Evaluation and Performance Analysis
- Feedback in Multi-User MIMO systems
- Role of Feedback in Ad-Hoc Networks