ANALYZING THE EFFECT OF CHANNEL ESTIMATION ERRORS ON THE AVERAGE BLOCK ERROR PROBABILITY OF A MISO TRANSMIT BEAMFORMING SYSTEM

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ABSTRACT

We address the problem of analyzing the effect of channel estimation errors on the average block error probability (BLEP) of transmit beamforming multiple input single output (MISO) system in a slowly varying Rayleigh fading wireless channel. For this purpose, we develop an accurate model of estimation errors in a block fading context and derive an analytical expression quantifying the impact of channel estimation errors on feedback MISO systems. In particular, the block fading model implies that the channel estimate and its error component are fixed for the entire block and an appropriate performance criteria is the average BLEP, a more complex metric to analyze. The derived closed form analytical expression for the average BLEP is validated by simulations.

Index Terms: MISO systems, transmit beamforming, channel estimation errors, average block error probability

1. INTRODUCTION

The performance of feedback based multiple input single output systems suffer from many forms of imperfections. The most common sources of imperfection are channel estimation errors, feedback delay and finite-rate channel quantization. An information theoretic approach to transmit beamforming with imperfect feedback is presented in [1] and [2].

In our previous work we studied the effect of all three forms of imperfection on the average symbol and bit error probabilities of various constellations [3]- [5]. In this paper, we take a closer look at the imperfections and modeling assumptions. For feedback systems to be practically viable, the coherence time of the channel variations has to be comparable to the feedback delay. An important consequence of the slow channel variation assumption is that averaging over the channel variations within a single block is no longer appropriate leading to the consideration of block error probabilities (BLEP). To the best of our knowledge, the effect of feedback imperfections on the average block error probability, an important and a meaningful system metric, has not received much attention. Due to space limitations, in this paper we only consider the effects of channel estimation errors on the average BLEP. As will be evident from the results in the paper, conceptually and analytically, handling the estimation errors for block fading channel model is quite involved and is not a simple extension of the results from average symbol and bit error probabilities.

Due to the presence of thermal noise, channel estimation errors are inevitable in any practical system. It is now a common practice to model the channel and its estimate as a jointly Gaussian random process, with an error term that is orthogonal to the channel estimate [2]- [6] (and the references *therein*). The error term associated with a particular channel estimate is unknown to the receiver and hence it becomes part of noise when the performance analysis is carried out. In a block fading model, the channel (and the estimation error) is assumed to remain constant for the entire block. In this paper, we also follow the standard model of joint Gaussianity between the channel and its estimate but adapt it to the block fading model.

If the channel under consideration is varying at symbol level, or if the performance criteria is average symbol/bit error probability, then the variance of the error term will be simply added (along with the symbol dependency) to the variance of the receiver noise resulting in an effective noise term with variance equaling the sum of variance of receiver noise and the variance of the estimation error term [2]- [6]. In a block fading model, the average BLEP analysis has to consider the fact that error term is constant for the entire block while each symbol experiences a different noise sample.

In summary, the contributions of this paper are twofold; one is accurate modeling of estimation errors in a block fading context and the other is deriving an analytical expression quantifying the impact of estimation errors on average block error probability. Both these contributions have much general applicability and are of general interest. The rest of this paper is organized as follows. In Section 2, we present the system model. Analytical expression for the average BELP with BPSK constellation is derived in Section 3. Numerical and simulation results are presented in Section 4. We conclude this paper in Section 5.

Notation: Small and upper case bold letters indicate vector and matrix respectively. $E(.)$, $(.)^T$, $(.)^H$, $|.|$, $\bar{(.)}$, and $\|.\|$ denote expectation, transpose, Hermitian, absolute value, complex conjugate, and norm respectively. $\mathbf{x} \sim \mathcal{NC}(\mu, \mathbf{\Sigma})$ indicates a circularly symmetric complex Gaussian (CSCG) random variable $\mathbf{x}$ with mean μ and covariance $\mathbf{\Sigma}$.

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2. SYSTEM MODEL

We consider a MISO system with t antennas at the base station (BS) and one antenna at the mobile station (MS). Let $\mathbf{h}_\ell$ be the channel between the BS and the MS for the ℓ^{th} block. $\mathbf{h}_\ell$ is modeled as a spatially i.i.d frequency-flat Rayleigh fading channel which is constant for all the N number of uncoded symbols in block ℓ. The vector valued channel $\mathbf{h}_\ell \sim \mathcal{NC}(\mathbf{0}, \mathbf{I})$. The transmitted k^{th} symbol in the block ℓ is denoted by $s_{m,\ell}[k]$ and $E[|s_{m,\ell}[k]|^2] = E_s$. Let $\mathbf{w}_\ell$ be the unit norm beamforming vector (BV) at the BS for the block ℓ. Then, the k^{th} received signal in the block ℓ is given by

$$y_\ell[k] = \mathbf{h}_\ell^H \mathbf{w}_\ell s_{m,\ell}[k] + \eta_\ell[k], \quad k = 1, 2, \cdots, N \tag{1}$$

where $\eta_\ell[k] \sim \mathcal{NC}\left(0, \sigma_n^2\right)$.

2.1. Channel Estimation Errors - Block Fading Model

Let $\widehat{\mathbf{h}}_\ell$ be the imperfectly estimated version of $\mathbf{h}_\ell$. We assume that $\mathbf{h}_\ell$ and $\widehat{\mathbf{h}}_\ell$ are jointly Gaussian, this assumption is well justified for many practical estimation techniques [2] and [6]. $\mathbf{h}_\ell$ and $\widehat{\mathbf{h}}_\ell$ can be related as follows:

$$\mathbf{h}_\ell = \frac{\tilde{\rho}_e}{\sqrt{\Lambda}} \widehat{\mathbf{h}}_\ell + \sqrt{1 - \rho_e^2}\, \varepsilon_\ell \tag{2}$$

where $\varepsilon_\ell \sim \mathcal{NC}(\mathbf{0}, \mathbf{I})$, $\widehat{\mathbf{h}}_\ell \sim \mathcal{NC}(\mathbf{0}, \Lambda\mathbf{I})$, and $\tilde{\rho}_e = \rho_e\, e^{j\phi_{\rho_e}}$ is the complex correlation coefficient that determines the degree of accuracy in channel estimation. With the help of training symbols $\tilde{\rho}_e$ can be assumed to be known at the receiver. Assuming instantaneous feedback and no channel quantization, the beamforming vector is given by

$$\mathbf{w}_\ell = \frac{\widehat{\mathbf{h}}_\ell}{\|\widehat{\mathbf{h}}_\ell\|}. \tag{3}$$

The k^{th} received signal of block ℓ with the BV given in (3) and $\mathbf{h}_\ell$ given in (2) is

$$\begin{aligned} y_\ell[k] &= \left(\frac{\tilde{\rho}_e}{\sqrt{\Lambda}} \widehat{\mathbf{h}}_\ell + \sqrt{1 - \rho_e^2} \varepsilon_\ell\right)^H \left(\widehat{\mathbf{h}}_\ell / \|\widehat{\mathbf{h}}_\ell\|\right) s_{m,\ell}[k] + \eta_\ell[k], \\ &= \left(\frac{\bar{\tilde{\rho}}_e}{\sqrt{\Lambda}} \|\widehat{\mathbf{h}}_\ell\| + \sqrt{1 - \rho_e^2} \tilde{\varepsilon}_\ell\right) s_{m,\ell}[k] + \eta_\ell[k]. \end{aligned} \tag{4}$$

In the above equation $\tilde{\varepsilon}_\ell$ is unknown to the receiver. For M-PSK constellation the signal information is in the phase. Since $\tilde{\varepsilon}_\ell \sim \mathcal{NC}(0, 1)$ is a CSCG random variable the phase can be absorbed into $\tilde{\varepsilon}_\ell$ without effecting its distribution.

In our previous work [3]- [5], our performance metric was symbol/bit error probability, as a consequence the estimation error term simply became part of receiver noise increasing its effective variance. This however is not appropriate in the block fading context (with average BLEP being the performance metric), an assumption necessary for feedback systems to be practically viable. In the block fading model the estimation error related term $\tilde{\varepsilon}_\ell$ is constant for the entire block (while each symbol experiences a different noise sample). The impact of this assumption on performance is different leading to different conclusions and the analysis is also further complicated. Note that (4) can be modified to include the effects of feedback delay and channel quantization. Due to page limitations in this paper we restrict our attention to estimation errors only.

3. AVERAGE BLOCK ERROR PROBABILITY

In this section we derive the average block error probability of an un-coded block of N BPSK symbols. The k^{th} received signal of the ℓ^{th} block is given by (4). Since the receiver knows $\tilde{\rho}_e$, it can compensate for the phase rotation of the received signal and then uses the real part (because of BPSK) of the received signal to decode the transmitted symbol as

$$\begin{aligned} \tilde{y}_\ell[k] &= \text{Real}\left(e^{j\phi_{\rho_e}}\, y_\ell[k]\right) = z\, s_{m,\ell}[k] + \hat{\eta}_\ell[k], \\ z &= \frac{\rho_e}{\sqrt{\Lambda}} \|\widehat{\mathbf{h}}_\ell\| + \sqrt{\frac{1 - \rho_e^2}{2}}\, \hat{\varepsilon}_\ell, \; -\infty < z < \infty, \end{aligned} \tag{5}$$

$\hat{\varepsilon}_\ell \sim \mathcal{N}(0, 1)$ and $\hat{\eta}_\ell[k] \sim \mathcal{N}\left(0, \sigma_n^2/2\right)$, $\hat{\varepsilon}_\ell$ and $\hat{\eta}_\ell[k]$ are both real random variables. Note that $\hat{\varepsilon}_\ell$ is a fixed constant for a particular block, over a number of blocks it is a statistical quantity, where as the noise sample $\hat{\eta}_\ell[k]$ is different for each symbol in a particular block. The pdf of 'z', central to the performance analysis, is derived in Appendix-I and is given in (15). Conditioned on z, the block error probability (the probability that at least one symbol in the block is received incorrectly) is given by

$$\begin{aligned} P_{B,\ell}(\gamma_b, \rho_e, t, N) &= 1 - \left\{1 - p_{b,\ell}\right\}^N, \\ &= 1 - \sum_{m=0}^{N} \begin{pmatrix} N \\ m \end{pmatrix} (-1)^m\, (p_{b,\ell})^m, \end{aligned}$$

where $p_{b,\ell}$ is the error probability of a symbol in the ℓ^{th} block. Similar to [7], the above equation can be easily modified to the scenario of a block channel coded system. Note that since z is fixed for the entire block, all the symbols have the same error probability. The average block error probability is given by

$$\begin{aligned} \tilde{P}_B(\gamma_b, \rho_e, t, N) &= E_\ell\left[P_{B,\ell}(\gamma_b, \rho_e, t, N)\right], \\ &= 1 - \sum_{m=0}^{N} \begin{pmatrix} N \\ m \end{pmatrix} (-1)^m\, E_\ell\left[(p_{b,\ell})^m\right]. \end{aligned}$$

Accounting for the fact that 'z' can be negative, with BPSK constellation, $E_\ell\left[(p_{b,\ell})^m\right]$ can be written as (6) (shown at the top of next page), where Q is the standard Gaussian tail function, γ_b is the signal-to-noise ratio (SNR) per symbol, and A_p is the area of $p(z)$, $z > 0$. Closed-form expression for A_p is derived in Appendix-II. Note that in the evaluation of $E_\ell\left[(p_{b,\ell})^m\right]$, one can integrate w.r.t z directly, however, for clarity in presentation we chose to express $E_\ell\left[(p_{b,\ell})^m\right]$ as shown in (6). In (6),

$$z_1 = z^2, \quad 0 < z < \infty, \qquad z_2 = z^2, \quad -\infty < z < 0.$$

We now have to carry out the expectation with respect to the random variables z_1 and z_2, which we do next. Using transformation of random variables, the pdfs of z_1 and z_2 can be shown to be given by

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$$
\begin{aligned}
E_\ell\left[(p_{b,\ell})^m\right] &= A_p E_\ell\left[Q^m\left(\sqrt{2\gamma_b z_1}\right)\right] + (1-A_p)\,E_\ell\left\{1 - Q\left(\sqrt{2\gamma_b z_2}\right)\right\}^m, \\
&= A_p\,E_{z_1}\left[Q^m\left(\sqrt{2\gamma_b z_1}\right)\right] + (1-A_p)\sum_{w=0}^{m}(-1)^w\,E_{z_2}\left\{Q^w\left(\sqrt{2\gamma_b z_2}\right)\right\} \qquad (6)
\end{aligned}
$$

$$
p(z_1) = \frac{\mathcal{R}_1}{A_p}\,z_1^{-\frac{1}{2}}\,e^{-z_1\frac{2-\rho_e^2}{2(1-\rho_e^2)}}\,D_{-2t}\left(\frac{-\sqrt{2\,z_1}\,\rho_e}{\sqrt{1-\rho_e^2}}\right), \qquad (7)
$$

$$
p(z_2) = \frac{\mathcal{R}_1}{(1-A_p)}\,z_2^{-\frac{1}{2}}\,e^{-z_2\frac{2-\rho_e^2}{2(1-\rho_e^2)}}\,D_{-2t}\left(\frac{\sqrt{2\,z_2}\,\rho_e}{\sqrt{1-\rho_e^2}}\right) \qquad (8)
$$

where

$$
\mathcal{R}_1 = \frac{(1-\rho_e^2)^t\,\Gamma(2t)}{2^t\,\Gamma(t)\sqrt{\pi(1-\rho_e^2)}},
$$

and $D_p(l)$ is the parabolic cylinder function. The Parabolic cylinder function has many representations. Because of analytical simplicity we chose to work with the representation given in (9) (shown at top of next page), where ${}_1F_1(\cdot,\cdot;\cdot)$ is the confluent hypergeometric function of the first kind. Using the representation in (9) for the parabolic cylinder function, (7) and (8) can be written as

$$
\begin{aligned}
p(z_1) &= \hat{p}_1(z_1) + \hat{p}_2(z_1), \quad z_1 > 0, \\
p(z_2) &= \hat{p}_1(z_2) - \hat{p}_2(z_2), \quad z_2 > 0,
\end{aligned}
$$

$$
\hat{p}_1(x) = \mathcal{L}\,x^{-\frac{1}{2}}e^{-\frac{x}{1-\rho_e^2}}\,{}_1F_1\left(t, \frac{1}{2}; \frac{\rho_e^2 x}{1-\rho_e^2}\right),\ x > 0, \qquad (10)
$$

$$
\hat{p}_2(x) = \mathcal{L}_1\,e^{-\frac{x}{1-\rho_e^2}}\,{}_1F_1\left(t+\frac{1}{2}, \frac{3}{2}; \frac{\rho_e^2 x}{1-\rho_e^2}\right),\ x > 0, \qquad (11)
$$

$$
\mathcal{L} = \frac{(1-\rho_e^2)^{t-\frac{1}{2}}\,\Gamma(2t)}{\Gamma\left(t+\frac{1}{2}\right)\,\Gamma(t)2^{2t-1}}, \quad \mathcal{L}_1 = \frac{\rho_e\,(1-\rho_e^2)^{t-1}\,\Gamma(2t)}{\left[\Gamma\left(t\right)\right]^2\,2^{2t-1}}.
$$

We now evaluate $E_{z_1}\left[Q^m\left(\sqrt{2\,\gamma_b\,z_1}\right)\right],\ m = 1, 2, \ldots, N,$ required to compute (6).

$$
\begin{aligned}
E_{z_1}\left[Q^m\left(\sqrt{2\,\gamma_b\,z_1}\right)\right] &= \int_{z_1=0}^{\infty} Q^m\left(\sqrt{2\,\gamma_b\,z_1}\right)p(z_1)\,dz_1, \\
E_{z_1}\left[Q\left(\sqrt{2\,\gamma_b\,z_1}\right)\right] &= \mathcal{G}_1\left(\frac{\pi}{2}\right) + \mathcal{G}_2\left(\frac{\pi}{2}\right), \qquad (12)
\end{aligned}
$$

where $\mathcal{G}_1(\varphi)$ and $\mathcal{G}_2(\varphi)$ are derived in Appendix-III. For $m = 1$ and 2, in Appendix-III we exploit the fact that the first and second powers of Gaussian Q-function are parameterized by $\varphi = \frac{\pi}{2}$ and $\varphi = \frac{\pi}{4}$ respectively in the function $\tilde{Q}(x)$ given below [9].

$$
\tilde{Q}(x) = \frac{1}{\pi}\int_{\theta=0}^{\varphi} e^{\left(-\frac{x^2}{2\sin^2\theta}\right)}\,d\theta,\ x \geq 0. \qquad (13)
$$

Following the above steps and Appendix-III it is easy to show that

$$
\begin{aligned}
E_{z_1}\left\{Q^2\left(\sqrt{2\gamma_b z_1}\right)\right\} &= \mathcal{G}_1\left(\frac{\pi}{4}\right) + \mathcal{G}_2\left(\frac{\pi}{4}\right), \\
E_{z_2}\left\{Q\left(\sqrt{2\gamma_b z_2}\right)\right\} &= \mathcal{G}_1\left(\frac{\pi}{2}\right) - \mathcal{G}_2\left(\frac{\pi}{2}\right), \\
E_{z_2}\left\{Q^2\left(\sqrt{2\gamma_b z_2}\right)\right\} &= \mathcal{G}_1\left(\frac{\pi}{4}\right) - \mathcal{G}_2\left(\frac{\pi}{4}\right).
\end{aligned}
$$

For $m \geq 3$, exact expressions are difficult to derive. We use the following approximation [8] for $Q^m(x)$ to derive expressions for $E\left[Q^m(x)\right]$ when $m \geq 3$.

$$
Q^m(x) \approx \sum_{k_1,k_2,\ldots,k_{m_a}} K_M\,C_M\,x^{f_m}\,e^{\frac{-mx^2}{2}}, \qquad (14)
$$

where the summation is taken over all sequences of nonnegative integer indices $k_1, \ldots, k_{m_a}$ such that $k_1 + \cdots + k_{m_a} = m$. In (14),

$$
\begin{aligned}
K_M &= \frac{m!}{(k_1)!(k_2)!\ldots(k_{m_a})!}, \\
f_m &= k_2 + 2k_3 + \cdots + (m_a - 1)k_{m_a}, \\
C_M &= (c_1)^{k_1}(c_2)^{k_2}\ldots(c_{m_a})^{k_{m_a}}, \\
c_{\tilde{m}} &= \frac{(-1)^{\tilde{m}+1}(A)^{\tilde{m}}}{B\sqrt{\pi}\,(\sqrt{2})^{\tilde{m}+1}\,\tilde{m}!}
\end{aligned}
$$

where $A = 1.98$, $m_a = 8$, and $B = 1.135$ [8]. The expectations of the Q approximation with respect to z_1 and z_2 are carried out in Appendix-IV and the final expressions are given by (27) and (29).

4. SIMULATION RESULTS

In this section we present a sample simulation to verify the derived analytical expression for the average block error probability. Fig. 1 shows the accuracy of derived analytical expression for average block error probability of transmit beamforming with imperfect channel estimate of a block of $N \in \{30, 50\}$ BPSK symbols with $t \in \{2, 3\}$ antennas and $\rho_e \in \{0.98, 0.95\}$. As pointed out earlier, the first two powers of Q-function are evaluated exactly (in the form of infinite series). For higher powers ($m \geq 3$ in (6)), we calculate the expectation w.r.t to the tractable approximation of Q-function given in (14) (a finite series). Fig. 1 further validates the tightness of the approximation of Gaussian Q-function (14).

5. CONCLUSION

For the block fading channel model, we considered the problem of analyzing the performance of transmit beamforming MISO systems under a realistic scenario of feedback based on an imperfectly estimated channel. The block fading model

$$D_p(l) = 2^{\frac{p}{2}} e^{-\frac{l^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)} \, {}_1F_1\left(-\frac{p}{2}, \frac{1}{2}; \frac{l^2}{2}\right) - \frac{\sqrt{2\pi}\, l}{\Gamma\left(\frac{-p}{2}\right)} \, {}_1F_1\left(\frac{1-p}{2}, \frac{3}{2}; \frac{l^2}{2}\right) \right\}, \tag{9}$$

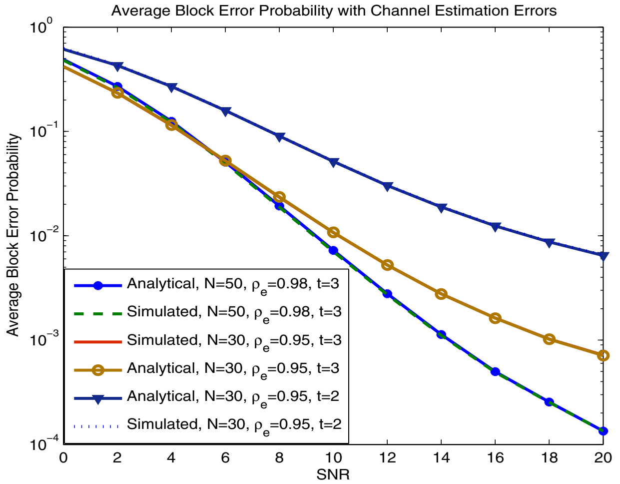

Fig. 1. Effect of channel estimation errors on the average block error probability.

implies that the channel estimate and its error component are fixed for the entire block, while the symbols in the block experience different noise samples. The performance criteria considered is the average BLEP of a block of un-coded BPSK symbols. The derived closed form analytical expression is validated by simulations.

Appendix-I

The signal scaling term z in (5) is

$$z = \frac{\rho_e}{\sqrt{\Lambda}} \|\widehat{\mathbf{h}}_\ell\| + \sqrt{(1-\rho_e^2)/2}\,\hat{\varepsilon}_\ell, \quad -\infty < z < \infty,$$

in the above equation, $\rho_e \|\widehat{\mathbf{h}}_\ell\|$ and $\hat{\varepsilon}_\ell$ are distributed as

$$\frac{\rho_e}{\sqrt{\Lambda}} \|\widehat{\mathbf{h}}_\ell\| \sim \frac{2x^{2t-1}}{\rho_e^{2t}\,\Gamma(t)} e^{\frac{-x^2}{\rho_e^2}}, \quad \hat{\varepsilon}_\ell \sim \mathcal{N}(0,1).$$

Conditioned on $\frac{\rho_e}{\sqrt{\Lambda}} \|\widehat{\mathbf{h}}_\ell\|$, the conditional pdf of z is given by

$$p(z|x) = \frac{1}{\sqrt{\pi(1-\rho_e^2)}} e^{-\frac{(z-x)^2}{1-\rho_e^2}}.$$

$$p(z) = \int_{-\infty}^{\infty} p(z,x)dx = \int_{-\infty}^{\infty} p(z|x)p(x)dx,$$

$$p(z) = \frac{1}{\rho_e^{2t}\,\Gamma(t)\sqrt{\pi(1-\rho_e^2)}} e^{-\frac{z^2}{1-\rho_e^2}} \int_0^{\infty} x^{2t-1} e^{\frac{-x^2}{\rho_e^2(1-\rho_e^2)} + \frac{2xz}{1-\rho_e^2}}.$$

With the help of (3.462-1) from [10], $p(z)$ can now be written as

$$p(z) = \mathcal{H} e^{-z^2\left(\frac{2-\rho_e^2}{2(1-\rho_e^2)}\right)} D_{-2t}\left(\frac{-\sqrt{2}\rho_e z}{\sqrt{1-\rho_e^2}}\right), \ -\infty < z < \infty, \tag{15}$$

where

$$\mathcal{H} = \frac{(1-\rho_e^2)^t\,\Gamma(2t)}{2^{t-1}\Gamma(t)\sqrt{\pi(1-\rho_e^2)}},$$

and $D_p(l)$ is the parabolic cylinder function [10].

Appendix-II

$$A_p = \int_0^{\infty} p(z)dz = 1 - \int_0^{\infty} p(-z)dz,$$

$$= 1 - \mathcal{H} \int_0^{\infty} e^{-z^2\left(\frac{2-\rho_e^2}{2(1-\rho_e^2)}\right)} D_{-2t}\left(\frac{\sqrt{2}\,\rho_e\, z}{\sqrt{1-\rho_e^2}}\right) dz.$$

Let $z = \sqrt{x}$, $dz = \frac{1}{2}x^{-\frac{1}{2}}dx$, the area A_p is

$$A_p = 1 - \frac{\mathcal{H}}{2} \int_0^{\infty} x^{-\frac{1}{2}} e^{-x\left(\frac{2-\rho_e^2}{2(1-\rho_e^2)}\right)} D_{-2t}\left(\frac{\sqrt{2}\,\rho_e\,\sqrt{x}}{\sqrt{1-\rho_e^2}}\right) dx,$$

$$= 1 - \frac{\Gamma(2t)(1-\rho_e^2)^t}{4^t\Gamma(t)\Gamma(t+1)} \, {}_2F_1\left(t, \frac{1}{2}; t+1; 1-\rho_e^2\right), \tag{18}$$

where ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is the hypergeometric function. To evaluate the above integral we used (7.725-6) from [10].

Appendix-III

Derivation of $\mathcal{G}_1(\varphi)$:

$$\mathcal{G}_1(\varphi) = \int_{\tilde{y}=0}^{\infty} \tilde{Q}\left(\sqrt{2\,\gamma_b\,\tilde{y}}\right) \hat{p}_1(\tilde{y})\, d\tilde{y}, \tag{19}$$

where $\tilde{Q}(x)$ is defined in (13), and$\hat{p}_1(x)$ is defined in (10). $\mathcal{G}_1(\varphi)$ can now be written as (16) (shown in the next page). With $\tilde{y} = x\left(1-\rho_e^2\right)/\rho_e^2$, (16) becomes

$$\mathcal{G}_1(\varphi) = \frac{\mathcal{L}\sqrt{1-\rho_e^2}}{\pi\,\rho_e} \int_{\theta=0}^{\varphi} d\theta \int_{x=0}^{\infty} x^{-\frac{1}{2}} e^{-z\mathcal{S}} \, {}_1F_1\left(t, \frac{1}{2}; x\right) dx,$$

where

$$\mathcal{S} = \frac{\gamma_b}{\sin^2\theta}\left(\frac{1-\rho_e^2}{\rho_e^2}\right) + \frac{1}{\rho_e^2}. \tag{20}$$

Notice that $\mathcal{S} > 1$ for $\rho_e < 1$. To evaluate the above equation, we use the identity given in (17) (shown in the next page). In the present context, in (17), $c = \frac{1}{2}$, $d = \mathcal{S}$, and $q = -t$.

$$\mathcal{G}_1(\varphi) = \frac{\mathcal{L}\,\Gamma\left(\frac{1}{2}\right)\sqrt{1-\rho_e^2}}{\pi\,\rho_e} \int_{\theta=0}^{\varphi} d^{-c}\left(1-d^{-1}\right)^q d\theta,$$

where $d^{-1} = \frac{\rho_e^2\sin^2\theta}{\sin^2\theta + c_1}$, and

$$c_1 = \gamma_b\,(1-\rho_e^2). \tag{21}$$

By using the generalized binomial series expansion,

$$\mathcal{G}_1(\varphi) \;=\; \frac{\mathcal{L}}{\pi} \int\limits_{\theta=0}^{\varphi} d\theta \int\limits_{\tilde{y}=0}^{\infty} \tilde{y}^{-\frac{1}{2}}\, e^{-\tilde{y}\left(\frac{\gamma_b}{\sin^2\theta}+\frac{1}{1-\rho_e^2}\right)}\, {}_1F_1\left(t, \frac{1}{2}; \frac{\rho_e^2\,\tilde{y}}{1-\rho_e^2}\right) d\tilde{y}. \tag{16}$$

$$\int\limits_{r=0}^{\infty} r^{c-1}\, e^{-r\,d}\, {}_1F_1\left(a, c; r\right) dr \;=\; \Gamma(c)\, d^{-c} \left(\frac{d-1}{d}\right)^{-a} = \Gamma(c)\, d^{-c}\, \left(1-d^{-1}\right)^{q},\ \operatorname{Re} c > 0,\, \operatorname{Re} d > 1. \tag{17}$$

$$\begin{aligned} (1-x)^q &= \sum_{n=0}^{\infty} \frac{(-1)^n\, \bar{p}_{k,n}}{n!}\, (x)^n\,, \\ \bar{p}_{k,n} &= q\,(q-1)\,\cdots\,(q-n+1). \end{aligned}$$

Notice that the convergence of the above series is not a problem since $d^{-1} < 1$. $\mathcal{G}_1(\varphi)$ can now be written in a series of steps from (22)- (25).

Derivation of $\mathcal{G}_2(\varphi)$:

$$\mathcal{G}_2(\varphi) \;=\; \int\limits_{\tilde{y}=0}^{\infty} \tilde{Q}\left(\sqrt{2\,\gamma_b\,\tilde{y}}\right)\, \hat{p}_2(\tilde{y})\, d\tilde{y}, \tag{30}$$

where $\hat{p}_2(\tilde{y})$, defined in (11). With $\tilde{y} = x\,\left(1-\rho_e^2\right)/\rho_e^2, \mathcal{G}_2(\varphi)$ can now be written as

$$\mathcal{G}_2(\varphi) = \frac{\mathcal{L}_1(1-\rho_e^2)}{\pi\rho_e^2} \int\limits_{\theta=0}^{\varphi} d\theta \int\limits_{x=0}^{\infty} e^{-x\mathcal{S}}\, {}_1F_1\left(t+\frac{1}{2}, \frac{3}{2}; x\right) dx,$$

$\mathcal{S}$ is defined in (20). $\mathcal{G}_2(\varphi)$ can be evaluated using the following series expansion for the Gauss hypergeometric function

$$\begin{aligned} {}_1F_1\left(a, b; r\right) dr &= \sum_{\bar{m}=0}^{\infty} \frac{a_{\bar{m}}\, r^{\bar{m}}}{b_{\bar{m}}\, \bar{m}!}, \\ a_{\bar{m}} &= 1.\, a_1\,(a_1+1)\cdots(a_1+\bar{m}-1), \qquad a_1 = t+\frac{1}{2}, \\ b_{\bar{m}} &= 1.\, b_1\,(b_1+1)\cdots(b_1+\bar{m}-1), \qquad b_1 = \frac{3}{2}. \end{aligned}$$

With the above series representation, $\mathcal{G}_2(\varphi)$ can now be written as

$$\begin{aligned} \mathcal{G}_2(\varphi) &= \frac{\mathcal{L}_1\,(1-\rho_e^2)}{\pi\,\rho_e^2} \sum_{\bar{m}=0}^{\infty} \frac{a_{\bar{m}}}{b_{\bar{m}}} \int\limits_{\theta=0}^{\varphi} d\theta \int\limits_{x=0}^{\infty} \frac{e^{-x\,\mathcal{S}}\, x^{\bar{m}}}{\bar{m}!}\, dx, \\ &= \frac{\mathcal{L}_1\,(1-\rho_e^2)}{\pi\,\rho_e^2} \sum_{\bar{m}=0}^{\infty} \frac{a_{\bar{m}}}{b_{\bar{m}}} \int\limits_{\theta=0}^{\varphi} \mathcal{S}^{-(\bar{m}+1)} d\theta, \\ &= \frac{\mathcal{L}_1\,(1-\rho_e^2)}{\pi} \sum_{\bar{m}=0}^{\infty} \frac{a_{\bar{m}}}{b_{\bar{m}}}\, \rho_e^{2\bar{m}}\, \mathcal{D}\left(\varphi, c_1, \bar{m}+1\right), \end{aligned} \tag{31}$$

where $\mathcal{D}\left(\varphi, c_1, \bar{m}+1\right)$ is defined in (26) and c_1 is defined in (21).

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$$\begin{aligned}\mathcal{G}_1(\varphi) &= \mathcal{M}\int\limits_{\theta=0}^{\varphi}\sum_{n=0}^{\infty}\frac{\rho_e^{2n}\,(-1)^n\,\bar{p}_{k,n}}{n!}\tilde{d}^{n+c}d\theta = \mathcal{M}\int\limits_{\theta=0}^{\varphi}\sum_{n=0}^{\infty}\frac{\rho_e^{2n}\,(-1)^n\,\bar{p}_{k,n}}{n!}\Big(1-(1-\tilde{d})\Big)^{n+c}d\theta, &(22)\\
&= \mathcal{M}\int\limits_{\theta=0}^{\varphi}\sum_{n=0}^{\infty}\frac{\rho_e^{2n}\,(-1)^n\,\bar{p}_{k,n}}{n!}\sum_{\bar{n}=0}^{\infty}\frac{(-1)^{\bar{n}}\,\bar{b}_{k,\bar{n}}}{\bar{n}!}\Big(1-\tilde{d}\Big)^{\bar{n}}d\theta, &(23)\\
&= \mathcal{M}\int\limits_{\theta=0}^{\varphi}\sum_{n=0}^{\infty}\frac{\rho_e^{2n}\,(-1)^n\,\bar{p}_{k,n}}{n!}\sum_{\bar{n}=0}^{\infty}\frac{(-1)^{\bar{n}}\,\bar{b}_{k,\bar{n}}}{\bar{n}!}\sum_{\bar{n}_1=0}^{\bar{n}}(-1)^{\bar{n}_1}\begin{pmatrix}\bar{n}\\ \bar{n}_1\end{pmatrix}\tilde{d}^{\,\bar{n}_1}d\theta, &(24)\\
&= \mathcal{M}\sum_{n=0}^{\infty}\frac{\rho_e^{2n}\,(-1)^n\,\bar{p}_{k,n}}{n!}\sum_{\bar{n}=0}^{\infty}\frac{(-1)^{\bar{n}}\,\bar{b}_{k,\bar{n}}}{\bar{n}!}\sum_{\bar{n}_1=0}^{\bar{n}}(-1)^{\bar{n}_1}\begin{pmatrix}\bar{n}\\ \bar{n}_1\end{pmatrix}\mathcal{D}\left(\varphi,c_1,\bar{n}_1\right), &(25)\end{aligned}$$

where

$$\bar{b}_{k,\bar{n}} = (n+c)\;(n+c-1)\cdots(n+c-\bar{n}+1)\,,\qquad \mathcal{M}=\frac{(1-\rho_e^2)^t\,\Gamma(2t)}{\Gamma\left(t+\frac{1}{2}\right)\,\Gamma(t)\,2^{2t-1}\,\sqrt{\pi}},\qquad \tilde{d}=\frac{\sin^2\theta}{\sin^2\theta+c_1},$$

$$\begin{aligned}\mathcal{D}\left(\varphi,c_1,\bar{n}_1\right) &= \int\limits_{\theta=0}^{\varphi}\tilde{d}\,d\theta = \int\limits_{\theta=0}^{\varphi}\left(\frac{\sin^2\theta}{\sin^2\theta+c_1}\right)^{\bar{n}_1}d\theta = \pi\Bigg\{\frac{\varphi}{\pi}-\frac{T}{\pi}\sqrt{\frac{c_1}{1+c_1}}\sum_{k=0}^{\bar{n}_1-1}\begin{pmatrix}2k\\ k\end{pmatrix}\frac{1}{[4(1+c_1)]^k}\\
&-\frac{2}{\pi}\sqrt{\frac{c_1}{1+c_1}}\sum_{k=0}^{\bar{n}_1-1}\sum_{j=0}^{k-1}\begin{pmatrix}2k\\ j\end{pmatrix}\frac{(-1)^{j+k}}{[4(1+c_1)]^k}\frac{\sin[(2k-2j)T]}{2k-2j}\Bigg\},\;0\leq\;\varphi\leq 2\pi, &(26)\end{aligned}$$

where

$$T = \frac{1}{2}\tan^{-1}\left(\frac{2\sqrt{c_1(1+c_1)}\sin 2\varphi}{(1+2\,c_1)\cos 2\varphi-1}\right)+\frac{\pi}{2}\left[1-2\sqrt{c_1(1+c_1)}\sin 2\varphi\left(\frac{(1+2\,c_1)\cos 2\varphi}{2}\right)\right].$$

Appendix-IV

$$E\left[Q^m\left(\sqrt{2\,\gamma_b\,z_1}\right)\right] = \int\limits_{z_1=0}^{\infty}Q^m\left(\sqrt{2\,\gamma_b\,z_1}\right)p(z_1)\,dz_1 = \int\limits_{z_1=0}^{\infty}Q^m\left(\sqrt{2\,\gamma_b\,z_1}\right)\left[\hat{p}_1(z_1)+\hat{p}_2(z_1)\right]\,dz_1,$$

where $\hat{p}_1(z_1)$ and $\hat{p}_2(z_2)$ are defined in (10) and (11) respectively. $E\left[Q^m\left(\sqrt{2\,\gamma_b\,z_1}\right)\right]$ can now be written as

$$\approx\sum_{k_1,k_2,\ldots,k_{m_a}}K_M\,C_M\int\limits_{z_1=0}^{\infty}\left[\mathcal{L}\,z_1^{-\frac{1}{2}}\;{}_1F_1\left(t,\frac{1}{2};\frac{\rho_e^2\,z_1}{1-\rho_e^2}\right)+\;{}_1F_1\left(t+\frac{1}{2},\frac{3}{2};\frac{\rho_e^2\,z_1}{1-\rho_e^2}\right)\mathcal{L}_1\right]\;z_1^{f_m}\,e^{-\left(m\gamma_b+\frac{1}{1-\rho_e^2}\right)\,z_1}\;dz_1,$$

$$E\left[Q^m\left(\sqrt{2\,\gamma_b\,z_1}\right)\right]\approx\sum_{k_1,k_2,\ldots,k_{m_a}}K_M\,C_M\left[\mathcal{L}\,\mathcal{F}\left(t,\frac{1}{2},f_m+\frac{1}{2},k_1,\,s_1\right)+\mathcal{L}_1\,\mathcal{F}\left(t+\frac{1}{2},\frac{3}{2},f_m+1,k_1,\,s_1\right)\right], \tag{27}$$

where $k_1=\rho_e^2/(1-\rho_e^2)$, $s_1=m\gamma_b+1/(1-\rho_e^2)$ and

$$\mathcal{F}\left(a,b,\alpha,k,s\right)=\int\limits_{x=0}^{\infty}x^{\alpha-1}e^{-s\,x}\;{}_1F_1\left(a,b;k\,x\right)\,dx=\Gamma(\alpha)\,s^{-\alpha}\;{}_2F_1\left(a,\alpha\,;b\,;ks^{-1}\right),\quad[\alpha>0,\,|s|>|k|]\,. \tag{28}$$

Following same steps as above we can write

$$E_{z_2}\left[Q^m\left(\sqrt{2\,\gamma_b\,z_2}\right)\right]\approx\sum_{k_1,k_2,\ldots,k_{m_a}}K_M\,C_M\left[\mathcal{L}\,\mathcal{F}\left(t,\frac{1}{2},f_m+\frac{1}{2},k_1,\,s_1\right)-\mathcal{L}_1\,\mathcal{F}\left(t+\frac{1}{2},\frac{3}{2},f_m+1,k_1,\,s_1\right)\right]. \tag{29}$$