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Analysis of Vector Quantizers Using Transformed Codebooks with Application to Feedback-Based Multiple Antenna Systems

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Abstract

Transformed codebooks are often obtained by a transformation of a given codebook, potentially optimum for a particular set of statistical conditions, to best match the statistical environment at hand. The procedure, though suboptimal, has recently been suggested for CSI feedback-based multiple antenna systems with correlated channels because of their simplicity and effectiveness. In this paper, we first consider the general distortion analysis of vector quantizers with transformed codebooks. Bounds on the average system distortion of this class of quantizers are provided. It exposes the effects of two kinds of sub-optimality introduced by the transformed codebook on system performance, which include the sub-optimal point density loss and the mismatched Voronoi shape. We then focus our attention on the application of the proposed general framework to providing capacity analysis of a feedback-based MISO system over spatially correlated fading channels. In particular, with capacity loss as an objective function, upper and lower bounds on the average distortion of MISO systems with transformed codebooks are provided and compared to that of the optimal channel quantizers. The expressions are examined to provide interesting insights in the high and low SNR regime. Numerical and simulation results are presented which confirm the tightness of the distortion bounds.

Index Terms

MIMO Channel Quantization, Finite-Rate Feedback, Transformed Codebook, Partial CSIT, Capacity Analysis, Vector Quantization, High-Resolution Distortion Analysis

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I. INTRODUCTION

This paper considers MIMO systems with partial channel state information (CSI) available at the transmitter, where CSI is conveyed from the receiver through a finite-rate feedback link. Recently, several interesting papers have appeared, proposing design algorithms as well as analytically quantifying the performance of finite-rate feedback multiple antenna systems [1] - [17]. We briefly discuss some of them below to provide context to this work.

Mukkavilli et. al. in [1] approximated the channel quantization region corresponding to each code point based on the channel geometric property and derived a universal lower bound on the outage probability of quantized MISO beamforming systems with arbitrary number of transmit antennas t over i.i.d. Rayleigh fading channels. Love and Heath in [2] and [3] related the problem to that of Grassmannian line packing [4]. Results on the density of Grassmannian line packings were derived and used to develop bounds on the codebook size given a capacity or SNR loss. Xia et. al. in [5] [6], Zhou et. al. in [7] and Roh and Rao et. al. in [8] [9] approximated the statistical distribution of the key random variable that characterizes the system performance. The distribution was used to analyze the performance of MISO systems with limited-rate feedback in the case of i.i.d. Rayleigh fading channels, and closed-form expressions of the capacity loss (or SNR loss) in terms of feedback rate B and the number of antennas t were obtained. Moreover, Roh et. al. [11] [12] extended the results from MISO channels to the case of MIMO systems with quantized feedback. Narula et. al. in [13] related the quantization problem to rate distortion theory, and obtained an approximation to the expected loss of the received SNR due to finite-rate quantization of the beamforming vectors in an MISO system with a large number of antennas t.

The analysis of finite-rate feedback systems has proven to be difficult and all the aforementioned approaches are case specific, limited to i.i.d. channels, mainly MISO channels, and are hard to extend to more complicated schemes. Recently, in [18] [19] we developed a general framework for the analysis of quantized feedback multiple antenna systems using a source coding perspective. The MIMO channel quantization was formulated as a general finite-rate vector quantization problem with attributes tailored to meet the general issues that arise in feedback-based communication systems, which include encoder side information, source vectors with constrained parameterizations, and general non-mean-squared distortion functions. Asymptotic distortion analysis of the proposed general quantization problem was provided by extending Bennett's classic analysis [20] as well as its corresponding vector extensions [21] [22]. By utilizing the proposed general framework, performance analysis of a finite-rate feedback MISO beamforming system transmitting over spatially correlated Rayleigh flat fading channels using both optimal and sub-optimal mismatched channel quantizers was provided in [23] [24].

The general framework developed in [18] [19] is versatile and has the potential for being adapted to deal with a variety of problems. The methodology, with suitable modifications, is used in this paper to enable the distortion analysis of a wide class of vector quantizers with transformed codebooks. *Transformed codebooks* are often obtained by a transformation of a given codebook, potentially optimum for a particular set of statistical conditions, to best match the statistical environment at hand. The procedure, though suboptimal, has recently been suggested for CSI feedback-based multiple antenna systems because of their simplicity and effectiveness. Love et. al. in [14] and Xia et. al. in [6] proposed a beamforming codebook design algorithm for correlated MIMO fading channels using a rotation-based transformation on the codebooks of the beamforming vectors, originally designed for i.i.d. fading channels. The rotation is derived from the channel correlation matrix. However, to the authors' knowledge, limited analytical results are available characterizing the performance of transformed channel quantizers for multiple antenna systems with finite-rate feedback.

In this paper, we focus our attention on investigating the effects of codebook transformation on the performance of multiple antenna systems with finite-rate CSI feedback. The contributions of this paper are twofold. We first consider the general problem of analyzing a vector quantizer with transformed codebook. Bounds on the average system distortion of this class of quantizers are provided. It exposes the effects of two kinds of sub-optimality introduced by the transformed codebook on system performance, which include the sub-optimal point density loss and the mismatched Voronoi shape. We then focus our attention on the application of the proposed general framework to providing capacity analysis of a feedback-based MISO system with spatially correlated fading channels using channel quantizers with transformed codebooks. In particular, using system capacity as the objective function, upper and lower bounds on the average distortion of MISO systems with transformed codebooks are provided and compared to that of the optimal channel quantizers. It is shown that the average distortion of CSI quantizers with transformed codebooks can be upper and lower bounded by a scaling of the distortion of optimal quantizers. Furthermore, based on numerical and simulation results, the scaling factors are shown to be close to one for fading channels whose channel covariance matrix has small to moderate condition numbers.

II. Background Information on the Generalized Vector Quantizer

Multiple antenna system with finite-rate CSI feedback was formulated as a generalized fixed-rate vector quantization problem in [18] [19] and analyzed by adapting tools from high resolution quantization theory. In this section, we briefly summarize the asymptotic distortion analysis of the generalized vector quantizer to facilitate understanding of the extension. Extension of the distortion analysis to quantizers with transformed codebook and its application to CSI-quantized MISO systems are provided in Section III - Section V.

A. General Vector Quantization Framework

It is assumed that the source variable $\mathbf{x} = (\mathbf{y},\, \mathbf{z})$ is a two-vector tuple with $\mathbf{y} \in \mathbb{Q}$ representing the actual quantization variable of dimension k_{q} and $\mathbf{z} \in \mathbb{Z}$ being the additional side information of dimension k_{z}. The *encoder side information* $\mathbf{z}$ is available at the encoder but not at the decoder. Quantization variable $\mathbf{y}$ and side information $\mathbf{z}$ have joint probability density function given by $p\,(\mathbf{y}, \mathbf{z})$. This paper considers a fixed-rate (B bits) quantizer with $N = 2^B$ quantization levels. Based on a particular source realization $\mathbf{x}$, the encoder (or the quantizer) represents vector $\mathbf{y}$ by one of the N vectors $\widehat{\mathbf{y}}_1, \widehat{\mathbf{y}}_2, \cdots, \widehat{\mathbf{y}}_N$, which form the codebook. The encoding or the quantization process is denoted as $\widehat{\mathbf{y}} = \mathcal{Q}(\mathbf{y},\, \mathbf{z})$. The distortion of a finite-rate quantizer is defined as $D = E_{\mathbf{x}}\Big[D_{\mathrm{Q}}\big(\mathbf{y}, \widehat{\mathbf{y}}\,;\, \mathbf{z}\big)\Big]$, where $D_{\mathrm{Q}}\big(\mathbf{y}, \widehat{\mathbf{y}}\,;\, \mathbf{z}\big)$ is a general *non-mean-squared distortion* function between $\mathbf{y}$ and $\widehat{\mathbf{y}}$ that is parameterized by $\mathbf{z}$. It is further assumed that function D_{Q} has a continuous second order derivative (or Hessian matrix w.r.t. to $\mathbf{y}$) $\mathbf{W}_{\mathbf{z}}(\widehat{\mathbf{y}})$ with the $(i, j)^{\text{th}}$ element given by

$$w_{i,j} = \frac{1}{2} \cdot \frac{\partial^2}{\partial\, y_i \partial\, y_j} D_{\mathrm{Q}}\big(\,\mathbf{y}\,, \widehat{\mathbf{y}}\,;\, \mathbf{z}\big)\Big|_{\mathbf{y}=\widehat{\mathbf{y}}} \quad . \tag{1}$$

B. Asymptotic Distortion Integral of the General Vector Quantizer

Under high resolution assumptions (large N), the asymptotic distortion of a finite-rate feedback system has been shown to have the following form, which is similar to Bennett's integral provided in [20] and its vector extension provided in [21],

$$D = E\Big[D_{\mathrm{Q}}\big(\,\mathbf{y}\,, Q(\mathbf{y}, \mathbf{z})\,;\, \mathbf{z}\big)\Big] = 2^{-\frac{2B}{k_{\mathrm{q}}}} \int_{\mathbb{Z}} \int_{\mathbb{Q}} I\big(\mathbf{y}\,;\, \mathbf{z}\,;\, \mathbb{E}_{\mathbf{z}}(\mathbf{y})\big) p\big(\mathbf{y},\, \mathbf{z}\big) \lambda\big(\mathbf{y}\big)^{-\frac{2}{k_{\mathrm{q}}}}\; d\,\mathbf{y}\, d\,\mathbf{z}. \tag{2}$$

$\mathbb{E}_{\mathbf{z}}(\mathbf{y})$ denotes the asymptotic (as N approaches infinity) projected Voronoi cell that contains $\mathbf{y}$ with side information $\mathbf{z}$ and captures the shape attribute of the quantization cell. In equation (2), $\lambda(\mathbf{y})$, the point density, is a function representing the relative density of the codepoints such that $\lambda(\mathbf{y})\, d\,\mathbf{y}$ is approximately the fraction of quantization points in a small neighborhood of $\mathbf{y}$. Function $I\big(\mathbf{y}\,;\, \mathbf{z}\,;\, \mathbb{E}\big)$ is the normalized inertial profile that represents the asymptotic normalized distortion or the relative distortion of the quantizer $\mathcal{Q}$ at position $\mathbf{y}$ conditioned on side information $\mathbf{z}$ with Voronoi shape $\mathbb{E}$. It is given by

$$I\big(\mathbf{y}\,;\, \mathbf{z}\,;\, \mathbb{E}\big) \triangleq \left(\int_{\mathbf{y}' \in \mathbb{E}} d\mathbf{y}'\right)^{-\frac{2+k_{\mathrm{q}}}{k_{\mathrm{q}}}} \cdot \left(\int_{\mathbf{y}' \in \mathbb{E}} \big(\mathbf{y}' - \mathbf{y}\big)^{\mathsf{T}} \cdot \mathbf{W}_{\mathbf{z}}(\mathbf{y}) \cdot \big(\mathbf{y}' - \mathbf{y}\big) \;\; d\,\mathbf{y}'\right) \quad . \tag{3}$$

The point density function $\lambda(\mathbf{y})$ and the normalized inertial profile $I\big(\mathbf{y}\,;\,\mathbf{z}\,;\,\mathbb{E}\big)$ are the key characteristics that can be used to describe the behavior of a specific quantizer. Alternately, given a vector quantizer, one has to find these two functions, as indicated in [19], and the average system distortion can then be obtained using (3).

C. Minimization of the Distortion Integral

The distortion integral given by equation (2) allows the minimization of the overall distortion by optimizing the choice of the Voronoi shape $\mathbb{E}_{\mathbf{z}}(\mathbf{y})$ and the point density function $\lambda(\mathbf{y})$. Firstly, the normalized inertial profile of an optimal quantizer can be defined as the minimum inertia of all admissible Voronoi regions (or shapes) $\mathbb{E}_{\mathbf{z}}(\mathbf{y})$, i.e.

$$I_{\text{opt}}\big(\mathbf{y}\,;\,\mathbf{z}\big) \overset{\triangle}{=} \min_{\mathbb{E}_{\mathbf{z}}(\mathbf{y})\in\mathcal{H}_{\text{Q}}} \; I\big(\mathbf{y}\,;\,\mathbf{z}\,;\,\mathbb{E}_{\mathbf{z}}(\mathbf{y})\big) \;\; , \tag{4}$$

where $\mathcal{H}_{\text{Q}}$ representing the set of all admissible tessellating polytopes that can tile space $\mathbb{Q}_{\mathbf{z}}$. It is known that finding the optimal Voronoi region as well as characterizing the exact optimal inertial profile is hard. However, the inertial profile of any Voronoi shape, including the optimal inertial profile, can be tightly lower bounded by that of an "M-shaped" hyper-ellipsoid with closed form expression given by

$$I\big(\mathbf{y}\,;\,\mathbf{z}\,;\,\mathbb{E}\big) \geq I_{\text{opt}}\big(\mathbf{y}\,;\,\mathbf{z}\big) \gtrapprox \widetilde{I}_{\text{opt}}\big(\mathbf{y}\,;\,\mathbf{z}\big) = \frac{k_{\text{q}}}{k_{\text{q}}+2}\cdot\left(\frac{\big|\mathbf{W}_{\mathbf{z}}(\mathbf{y})\big|}{\kappa_{k_{\text{q}}}^{2}}\right)^{\frac{1}{k_{\text{q}}}} , \qquad \kappa_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \;\; . \tag{5}$$

Secondly, by substituting the inertial profile lower bound (5) into the system distortion integral as well as utilizing the Holder's inequality to select the optimal point density, the asymptotic distortion of the generalized finite-rate quantization system can be lower bounded by $\widetilde{D}_{\text{Low}}$ given by

$$\widetilde{D}_{\text{Low}} = 2^{-\frac{2B}{k_{\text{q}}}} \cdot \left(\int_{\mathbb{Q}} \;\Big(\widetilde{I}^{\text{w}}_{\text{opt}}(\mathbf{y})\cdot p\big(\mathbf{y}\big)\Big)^{\frac{k_{\text{q}}}{2+k_{\text{q}}}} \; d\,\mathbf{y}\right)^{\frac{2+k_{\text{q}}}{k_{\text{q}}}} \quad , \tag{6}$$

where $\widetilde{I}^{\text{w}}_{\text{opt}}\big(\mathbf{y}\big)$ is the average optimal inertial profile defined as

$$\widetilde{I}^{\text{w}}_{\text{opt}}\big(\mathbf{y}\big) = \int_{\mathbb{Z}} \; \widetilde{I}_{\text{opt}}\big(\mathbf{y}\,;\,\mathbf{z}\big)\cdot p\big(\mathbf{z}\big|\,\mathbf{y}\big) \; d\,\mathbf{z} \;. \tag{7}$$

The optimal point density that minimizes the asymptotic system distortion is given by

$$\lambda^{*}\big(\mathbf{y}\big) = \Big(\widetilde{I}^{\text{w}}_{\text{opt}}\big(\mathbf{y}\big)\cdot p\big(\mathbf{y}\big)\Big)^{\frac{k_{\text{q}}}{2+k_{\text{q}}}} \cdot \left(\int_{\mathbb{Q}} \;\Big(\widetilde{I}^{\text{w}}_{\text{opt}}\big(\mathbf{y}\big)\cdot p\big(\mathbf{y}\big)\Big)^{\frac{k_{\text{q}}}{2+k_{\text{q}}}} \, d\,\mathbf{y}\right)^{-1} \;\; . \tag{8}$$

D. Distortion Analysis of Constrained Source

The analysis discussed above is for the case that the input source $\mathbf{y}$ is a free random vector of dimension k_{q}. In some situations, it is required to quantize the k_{q}-dimensional source vector $\mathbf{y} \in \mathbb{Q}$ subject to a multi-dimensional constraint function $\mathbf{g}(\mathbf{y}) = \mathbf{0}$ of size $k_{\text{c}} \times 1$. In this case, the proposed asymptotic distortion analysis has been shown to be still valid with some simple modifications. First, the degrees of freedom in $\mathbf{y}$ reduce from k_{q} to $k'_{\text{q}} = k_{\text{q}} - k_{\text{c}}$. Second, the sensitivity matrix is replaced by its constrained version $\mathbf{W}_{\text{c},\,\mathbf{z}}(\mathbf{y})$ given by

$$\mathbf{W}_{\text{c},\,\mathbf{z}}\big(\mathbf{y}\big) = \mathbf{V}_2^{\intercal} \cdot \mathbf{W}_{\mathbf{z}}\big(\mathbf{y}\big) \cdot \mathbf{V}_2 \ , \tag{9}$$

where $\mathbf{V}_2 \in \mathbb{R}^{k_{\text{q}} \times k'_{\text{q}}}$ is an orthonormal matrix with its columns constituting an orthonormal basis for the null space $\mathcal{N}\big(\frac{\partial}{\partial \mathbf{y}}\,\mathbf{g}(\mathbf{y})\big)$. Lastly, the multi-dimensional integrations used in evaluating the average distortions are over the constrained space $\mathbf{g}(\mathbf{y}) = 0$.

III. Asymptotic Distortion Analysis of Quantizers with Transformed Codebook

In certain situations, the underlying source distribution $p(\mathbf{y}, \mathbf{z})$ or the distortion function D_{Q} of the source variable varies during the quantization process. It is practically infeasible to design separate codebooks optimized for every different source distribution and distortion function, or the encoder and the decoder may not have the ability to store a large number of codebooks. In these situations, it is convenient to use a quantizer whose codebook is constructed by a transformation of a fixed codebook based on the current statistical distribution of the source variable. This type of quantizers are generally called transformed quantizers [25] [26], and have been used in conventional source coding area with a linear orthogonal transformation followed by a product quantizer. We provide in this subsection an analysis of the generalized vector quantizer, which is described in Section II, when a transformed codebook is used. Detailed applications to finite-rate feedback MISO systems with transformed codebook over spatially correlated fading channels are provided in Section V.

A. Problem Formulation

It is first assumed that all the codebooks are generated from one fixed codebook $\mathcal{C}_0$ which is designed to match source distribution $p_0(\mathbf{y},\,\mathbf{z})$, and distortion function $D_{0,\,\text{Q}}$ with sensitivity matrix $\mathbf{W}_{0,\,\mathbf{z}}(\mathbf{y})$. Codebook $\mathcal{C}_0$ has a point density given by $\lambda_0(\mathbf{y})$, and a normalized inertial profile $I_0\big(\mathbf{y};\mathbf{z};\mathbb{E}_{0,\mathbf{z}}(\mathbf{y})\big)$ that is optimized to match the distortion function $D_{0,\,\text{Q}}$, with $\mathbb{E}_{0,\mathbf{z}}(\mathbf{y})$ representing the asymptotic Voronoi cell that contains $\mathbf{y}$ with side information $\mathbf{z}$. Let the source distribution change to $p(\mathbf{y},\,\mathbf{z})$ from $p_0(\mathbf{y},\,\mathbf{z})$ and the distortion function become D_{Q} instead of $D_{0,\mathrm{Q}}$ with sensitivity matrix $\mathbf{W}_{\mathbf{z}}(\mathbf{y})$ instead of $\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})$. The encoder and decoder are assumed to adopt a transformed codebook $\mathcal{C}$ obtained from $\mathcal{C}_0$ by using a general one-to-one mapping $\mathbf{F}(\cdot)$ with both of its domain and codomain in space $\mathbb{Q}$, i.e.

$$\mathcal{C} = \left\{ \mathbf{F}(\widehat{\mathbf{y}}) \;\middle|\; \widehat{\mathbf{y}} \in \mathcal{C}_0 \right\} . \tag{10}$$

B. Sub-optimal Point Density & Sub-optimal Voronoi Shape

Assuming the codebook transformation function $\mathbf{F}(\cdot)$ has continuous first order derivative, two types of sub-optimality arise when the transformed quantizer is used. One comes from the sub-optimal point density $\lambda_{\text{tr}}(\mathbf{y})$, which can be derived from $\lambda_0(\mathbf{y})$ as

$$\lambda_{\text{tr}}(\mathbf{y}) = \frac{\lambda_0\left(\mathbf{F}^{-1}(\mathbf{y})\right)}{\left|\mathbf{F}_{\mathrm{d}}\left(\mathbf{F}^{-1}(\mathbf{y})\right)\right|} , \qquad \mathbf{F}_{\mathrm{d}}(\mathbf{y}) = \frac{\partial\,\mathbf{F}(\mathbf{y})}{\partial\,\mathbf{y}} \;\; . \tag{11}$$

If the source variable is subject to k_{c} constraints given by the vector equation $\mathbf{g}(\mathbf{y}) = \mathbf{0}$, the transformed point density is given by

$$\lambda_{\text{c-tr}}(\mathbf{y}) = \frac{\lambda_0\left(\mathbf{F}^{-1}(\mathbf{y})\right)}{\left|\mathbf{V}_2(\mathbf{y})^{\mathsf{T}} \cdot \mathbf{F}_{\mathrm{d}}\left(\mathbf{F}^{-1}(\mathbf{y})\right) \cdot \mathbf{V}_2\left(\mathbf{F}^{-1}(\mathbf{y})\right)\right|} \;\; , \tag{12}$$

where $\mathbf{V}_2(\mathbf{y})$ is an orthonormal matrix with its columns constituting an orthonormal basis of the null space $\mathcal{N}\left(\frac{\partial}{\partial\mathbf{y}}\,\mathbf{g}(\mathbf{y})\right)$. Compared to the optimal point density $\lambda^*(\mathbf{y})$ given by equation (8) which corresponds the optimally designed codebook, $\lambda_{\text{tr}}(\mathbf{y})$ given by equation (11) is always sub-optimal and hence leads to performance degradation. The other sub-optimality arises from the fixed location of the code points in the transformed codebook $\mathcal{C}$, in the sense that the Voronoi shape of the transformed code is not matched to the distortion function D_{Q} and hence is not optimized to minimize the inertial profile. Note that these two sub-optimalities, named as point density loss and cell shape loss, were also discussed in [25] in the setting of the conventional product quantizers and further applied to study the distortion performance of conventional quantizers with transformed codebooks.

C. Characterizing the Inertial Profile of the Transformed Codebook

Unfortunately, the Voronoi region $\mathbb{E}_{\text{tr},\mathbf{z}}\left(\widehat{\mathbf{y}}'_i\right)$ of the transformed codebook, which is defined to be

$$\mathbb{E}_{\text{tr},\mathbf{z}}\left(\widehat{\mathbf{y}}'_i\right) \triangleq \left\{ \mathbf{y} \;\middle|\; D_{\mathrm{Q}}\left(\mathbf{y},\, \widehat{\mathbf{y}}'_i\,;\, \mathbf{z}\right) \le D_{\mathrm{Q}}\left(\mathbf{y},\, \widehat{\mathbf{y}}'_j\,;\, \mathbf{z}\right) , \quad \forall\; \widehat{\mathbf{y}}'_j \in \mathcal{C} \;\;\&\;\; \widehat{\mathbf{y}}'_j \neq \widehat{\mathbf{y}}'_i \right\} , \qquad \widehat{\mathbf{y}}'_i \in \mathcal{C} \;, \tag{13}$$

is hard to characterize and depends both on the transformation $\mathbf{F}$ as well as the distortion function D_{Q}. In order to characterize the effects of the transformed Voronoi shape on the system distortion, lower and upper bounds of the normalized inertial profile of the transformed code are provided. First, let us consider a sub-optimal quantizer $\mathcal{Q}_{\text{sub}}(\cdot)$ with transformed codebook $\mathcal{C}$ that uses a sub-optimal encoding process, given by

$$\widehat{\mathbf{y}} = \mathcal{Q}_{\text{sub}}\big(\mathbf{y}, \mathbf{z}\big) = \mathbf{F}\bigg(\mathcal{Q}\Big(\mathbf{F}^{-1}(\mathbf{y}) \, , \mathbf{z}\Big)\bigg) \ , \tag{14}$$

where $\mathcal{Q}(\cdot)$ is the optimal encoder that matches to distortion function $D_{0,\,\text{Q}}$. This sub-optimal encoder can be viewed as an extension of the "companding" model introduced by Bennett in [20] to the general vector quantization problem. It was originally used in conventional scalar quantizers, where the encoder is a combination of a monotonically increasing nonlinear mapping $E(x)$, the compressor, followed by a uniform quantizer; and the corresponding decoder is composed of a uniform decoder followed by an inverse mapping E^{-1}, the expander. In the case of the generalized vector quantizer discussed here, the Voronoi shape of the sub-optimal transformed encoder $\mathcal{Q}_{\text{sub}}$ can be analytically characterized as

$$\mathbb{E}_{\text{sub},\mathbf{z}}\big(\mathbf{F}(\mathbf{y})\big) = \Big\{\mathbf{F}(\mathbf{y}') \ \Big| \ \mathbf{y}' \in \mathbb{E}_{0,\mathbf{z}}(\mathbf{y})\Big\} \ , \tag{15}$$

where $\mathbb{E}_{0,\mathbf{z}}(\mathbf{y})$ is the optimal Voronoi shape of the original codebook $\mathcal{C}_0$ corresponding to distortion function $D_{0,\,\text{Q}}$. Due to the sub-optimality of encoder $\mathcal{Q}_{\text{sub}}$, the normalized inertial profile of the transformed Voronoi shape $\mathbb{E}_{\text{tr},\mathbf{z}}(\mathbf{y})$ is upper bounded by the inertial profile of $\mathbb{E}_{\text{sub},\mathbf{z}}(\mathbf{y})$ given by (15), but lower bounded the inertial profile of the optimal Voronoi shape $\mathbf{E}_{\mathbf{z}}(\mathbf{y})$ corresponding to distortion function D_{Q}.

Proposition 1: Under high resolution assumptions, the approximated inertial profile $\widetilde{I}_{\text{tr}}\left(\mathbf{F}(\widehat{\mathbf{y}})\,;\,\mathbf{z}\right)$ of a quantizer with transformed codebook can be upper and lower bounded by the following form,

$$\begin{aligned}\frac{k_{\text{q}}}{k_{\text{q}}+2}\cdot\left(\frac{\big|\mathbf{W}_{\mathbf{z}}\big(\mathbf{F}(\mathbf{y})\big)\big|}{\kappa_{k_{\text{q}}}^2}\right)^{\frac{1}{k_{\text{q}}}} \;&=\; \widetilde{I}_{\text{opt}}\left(\mathbf{F}(\mathbf{y})\,;\,\mathbf{z}\right) \;\overset{a}{\leq}\; \widetilde{I}_{\text{tr}}\left(\mathbf{F}(\mathbf{y})\,;\,\mathbf{z}\right) \;\overset{b}{\leq}\; \widetilde{I}_{\text{sub}}\left(\mathbf{F}(\mathbf{y})\,;\,\mathbf{z}\right)\\ &=\; \frac{\big|\mathbf{F}_{\text{d}}(\mathbf{y})\big|^{-\frac{2}{k_{\text{q}}}}}{k_{\text{q}}+2}\cdot\left(\frac{\big|\mathbf{W}_{0,\,\mathbf{z}}(\mathbf{y})\big|}{\kappa_{k_{\text{q}}}^2}\right)^{\frac{1}{k_{\text{q}}}}\cdot\,\mathbf{tr}\Big(\mathbf{W}_{0,\,\mathbf{z}}(\mathbf{y})^{-1}\cdot\mathbf{F}_{\text{d}}(\mathbf{y})^{\mathsf{T}}\cdot\mathbf{W}_{\mathbf{z}}\big(\mathbf{F}(\mathbf{y})\big)\cdot\mathbf{F}_{\text{d}}(\mathbf{y})\Big)\ .\end{aligned} \tag{16}$$

Furthermore, if the source variable is subject to k_{c} constraints given by the vector equation $\mathbf{g}(\mathbf{y}) = \mathbf{0}$, the constrained inertial profile $\widetilde{I}_{\text{c-tr}}\left(\mathbf{F}(\widehat{\mathbf{y}})\,;\,\mathbf{z}\right)$ can be similarly bounded by

$$\begin{aligned}\frac{k_{\text{q}}'}{k_{\text{q}}'+2}\cdot\left(\frac{\Big|\mathbf{V}_2\big(\mathbf{F}(\mathbf{y})\big)^{\mathsf{T}}\cdot\mathbf{W}_{\mathbf{z}}\big(\mathbf{F}(\mathbf{y})\big)\cdot\mathbf{V}_2\big(\mathbf{F}(\mathbf{y})\big)\Big|}{\kappa_{k_{\text{q}'}}^2}\right)^{\frac{1}{k_{\text{q}'}}} \;&=\; \widetilde{I}_{\text{c-opt}}\left(\mathbf{F}(\mathbf{y})\,;\,\mathbf{z}\right) \;\overset{a}{\leq}\; \widetilde{I}_{\text{c-tr}}\left(\mathbf{F}(\mathbf{y})\,;\,\mathbf{z}\right)\\ \overset{b}{\leq}\; \widetilde{I}_{\text{c-sub}}\left(\mathbf{F}(\mathbf{y})\,;\,\mathbf{z}\right) \;=\; \frac{\Big|\mathbf{V}_2\big(\mathbf{F}(\mathbf{y})\big)^{\mathsf{T}}\cdot\mathbf{F}_{\text{d}}(\mathbf{y})\cdot\mathbf{V}_2(\mathbf{y})\Big|^{-\frac{2}{k_{\text{q}}'}}}{k_{\text{q}}'+2}&\left(\frac{\Big|\mathbf{V}_2(\mathbf{y})^{\mathsf{T}}\cdot\mathbf{W}_{0,\,\mathbf{z}}(\mathbf{y})\cdot\mathbf{V}_2(\mathbf{y})\Big|}{\kappa_{k_{\text{q}}'}^2}\right)^{\frac{1}{k_{\text{q}}'}}\\ \cdot\,\mathbf{tr}\left(\Big(\mathbf{V}_2(\mathbf{y})^{\mathsf{T}}\cdot\mathbf{W}_{0,\,\mathbf{z}}(\mathbf{y})\cdot\mathbf{V}_2(\mathbf{y})\Big)^{-1}\cdot\mathbf{V}_2(\mathbf{y})^{\mathsf{T}}\cdot\mathbf{F}_{\text{d}}(\mathbf{y})^{\mathsf{T}}\right.&\left.\cdot\,\mathbf{W}_{\mathbf{z}}\big(\mathbf{F}(\mathbf{y})\big)\cdot\mathbf{F}_{\text{d}}(\mathbf{y})\cdot\mathbf{V}_2(\mathbf{y})\right)\ ,\end{aligned} \tag{17}$$

where $\mathbf{V}_2(\mathbf{y})$ is an orthonormal matrix with its columns constituting an orthonormal basis for the null space $\mathcal{N}\left(\frac{\partial}{\partial\,\mathbf{y}}\,\mathbf{g}(\mathbf{y})\right)$.

Proof: See Appendix A. ■

D. Distortion Integral of the Transformed Codebook

By substituting the transformed point density (11) and the bounds of the transformed inertial profile given by (16) into the distortion integration (2), we can upper and lower bound the asymptotic system distortion of a transformed quantizer by the following form

$$\begin{aligned}
&\widetilde{D}_{\text{tr-Low}} = 2^{-\frac{2B}{k_q}} \cdot \left(\int_{\mathbb{Z}} \int_{\mathbb{Q}} \widetilde{I}_{\text{opt}}\left(\mathbf{y}\,;\,\mathbf{z}\right) \cdot p\big(\mathbf{y},\,\mathbf{z}\big) \cdot \lambda_{\text{tr}}\big(\mathbf{y}\big)^{-\frac{2}{k_q}}\, d\mathbf{y}\, d\mathbf{z} \right) \\
&\le \widetilde{D}_{\text{tr}} = 2^{-\frac{2B}{k_q}} \cdot \left(\int_{\mathbb{Z}} \int_{\mathbb{Q}} \widetilde{I}_{\text{tr}}\left(\mathbf{y}\,;\,\mathbf{z}\right) \cdot p\big(\mathbf{y},\,\mathbf{z}\big) \cdot \lambda_{\text{tr}}\big(\mathbf{y}\big)^{-\frac{2}{k_q}}\, d\mathbf{y}\, d\mathbf{z} \right) \\
&\le \widetilde{D}_{\text{tr-Upp}} = 2^{-\frac{2B}{k_q}} \cdot \left(\int_{\mathbb{Z}} \int_{\mathbb{Q}} \widetilde{I}_{\text{sub}}\left(\mathbf{y}\,;\,\mathbf{z}\right) \cdot p\big(\mathbf{y},\,\mathbf{z}\big) \cdot \lambda_{\text{tr}}\big(\mathbf{y}\big)^{-\frac{2}{k_q}}\, d\mathbf{y}\, d\mathbf{z} \right) \quad .
\end{aligned} \tag{18}$$

Similarly, by substituting (12) and (17) into (2), the asymptotic distortion of a constrained quantizer with transformed codebook is bounded by

$$\begin{aligned}
&\widetilde{D}_{\text{c-tr-Low}} = 2^{-\frac{2B}{k_{q'}}} \cdot \left(\int_{\mathbb{Z}} \int_{\mathbb{Q}} \widetilde{I}_{\text{c-opt}}\left(\mathbf{y}\,;\,\mathbf{z}\right) \cdot p\big(\mathbf{y},\,\mathbf{z}\big) \cdot \lambda_{\text{c-tr}}\big(\mathbf{y}\big)^{-\frac{2}{k'_q}}\, d\mathbf{y}\, d\mathbf{z} \right) \\
&\le \widetilde{D}_{\text{c-tr}} = 2^{-\frac{2B}{k_{q'}}} \cdot \left(\int_{\mathbb{Z}} \int_{\mathbb{Q}} \widetilde{I}_{\text{c-tr}}\left(\mathbf{y}\,;\,\mathbf{z}\right) \cdot p\big(\mathbf{y},\,\mathbf{z}\big) \cdot \lambda_{\text{c-tr}}\big(\mathbf{y}\big)^{-\frac{2}{k'_q}}\, d\mathbf{y}\, d\mathbf{z} \right) \\
&\le \widetilde{D}_{\text{c-tr-Upp}} = 2^{-\frac{2B}{k'_q}} \cdot \left(\int_{\mathbb{Z}} \int_{\mathbb{Q}} \widetilde{I}_{\text{c-sub}}\left(\mathbf{y}\,;\,\mathbf{z}\right) \cdot p\big(\mathbf{y},\,\mathbf{z}\big) \cdot \lambda_{\text{c-tr}}\big(\mathbf{y}\big)^{-\frac{2}{k'_q}}\, d\mathbf{y}\, d\mathbf{z} \right) \quad .
\end{aligned} \tag{19}$$

Similar to conventional product transformed code [25], there exist trade-offs between the two sub-optimalities: point density loss and Voronoi shape loss. To be specific, it is always possible to find a transformation $\mathbf{F}(\cdot)$ such that the transformed point density $\lambda_{\text{tr}}(\mathbf{y})$ matches exactly the optimal point density $\lambda^*(\mathbf{y})$. However, by doing so, the transformation may cause severe "oblongitis" of the Voronoi shape in some cases, which will lead to significant increase in the normalized inertial profile. Therefore, a transformation that optimally balances two types of losses should be employed. This tradeoff is directly reflected in the distortion bound $\widetilde{D}_{\text{tr, Upp}}$ where both $\widetilde{I}_{\text{sub}}\left(\mathbf{y}\,;\,\mathbf{z}\right)$ and $\lambda_{\text{tr}}(\mathbf{y})$ in (18) depend on the transformation $\mathbf{F}(\cdot)$. So is distortion bound $\widetilde{D}_{\text{c-tr, Upp}}$ given by (19).

IV. Analysis of Optimal MISO CSI Quantizers

By utilizing the high-resolution distortion analysis described in Section II (provided in detail in [18] and [27]), this section provides a detailed investigation of the capacity loss of a finite-rate CSI-quantized MISO beamforming system over correlated fading channels.

A. System Model of MISO Systems with Finite-Rate Feedback

We consider a MISO system, with t transmit antennas and one receive antenna, signaling through a frequency flat fading channel. The channel model can be represented as

$$y = \mathbf{h}^{\mathsf{H}} \cdot \mathbf{x} + n \ , \tag{20}$$

where y is the received signal (scalar), n is the additive complex Gaussian noise with zero mean and unit variance, and $\mathbf{h}^{\mathsf{H}} \in \mathbb{C}^{1\times t}$ is the correlated[1] MISO channel response with distribution given by $\mathbf{h} \sim \mathcal{N}_{\mathsf{c}}\big(\mathbf{\Sigma}_{\mathrm{h}}\big)$. The transmitted signal vector $\mathbf{x}$ is normalized to have a power constraint given by $E\big[\|\mathbf{x}\|^2\big] = \rho$, with ρ representing the average signal to noise ratio at each receive antenna.

In this paper, the channel state information $\mathbf{h}$ is assumed to be perfectly known at the receiver but only partially available at the transmitter through a finite-rate feedback link of B bits per channel update between the transmitter and receiver. To be specific, a quantization codebook $\mathcal{C} = \big\{\widehat{\mathbf{v}}_1, \cdots, \widehat{\mathbf{v}}_N\big\}$, which is composed of unit-norm transmit beamforming vectors, is assumed known to both the receiver and the transmitter. Based on the channel realization $\mathbf{h}$, the receiver selects the best code point $\widehat{\mathbf{v}}$ from the codebook and sends the corresponding index back to the transmitter. At the transmitter, the unit-norm vector $\widehat{\mathbf{v}}$ is employed as the beamforming vector, i.e.

$$y = \langle \mathbf{h}, \widehat{\mathbf{v}} \rangle \cdot s + n = \|\mathbf{h}\| \cdot \langle \mathbf{v}, \widehat{\mathbf{v}} \rangle \cdot s + n \,, \qquad E\big[\,|s|^2\big] = \rho \ . \tag{21}$$

where $\mathbf{v}$ is the channel directional vector given by $\mathbf{v} = \mathbf{h}/\|\mathbf{h}\|$. The corresponding ergodic capacity or the maximum system mutual information rate of the quantized MISO beamforming system is given by

$$C_{\mathrm{Q}} = E\Big[\log_2\Big(1 + \rho \cdot \|\mathbf{h}\|^2 \cdot |\langle \mathbf{v}, \widehat{\mathbf{v}}|^2\Big)\Big] \ . \tag{22}$$

With perfect channel state information available at the transmitter, which corresponds to the case of infinite rate feedback $B = \infty$, it is optimal to choose $\mathbf{v} = \mathbf{h}/\|\mathbf{h}\|$ as the transmit beamforming vector, and the corresponding system ergodic capacity is given by

$$C_{\mathrm{p}} = E\Big[\log_2\Big(1 + \rho \cdot \|\mathbf{h}\|^2\Big)\Big] \ . \tag{23}$$

Therefore, the performance of a CSI-feedback-based MISO system can be characterized by the capacity loss C_{Loss} due to the finite-rate quantization of the transmit beamforming vectors, which is defined as the expectation of the instantaneous mutual information rate loss $C_{\mathrm{L}}(\mathbf{h}, \widehat{\mathbf{v}})$, i.e.

$$C_{\mathrm{Loss}} = C_{\mathrm{p}} - C_{\mathrm{Q}} = E\big[\,C_{\mathrm{L}}(\mathbf{h}, \widehat{\mathbf{v}})\,\big] \,, \qquad C_{\mathrm{L}}(\mathbf{h}, \widehat{\mathbf{v}}) = -\log_2\left(1 - \frac{\rho \cdot \|\mathbf{h}\|^2}{1 + \rho \cdot \|\mathbf{h}\|^2} \cdot \Big(1 - |\langle \mathbf{v}, \widehat{\mathbf{v}} \rangle|^2\Big)\right) \ . \tag{24}$$

[1]For the sake of fair comparisons, we normalize the channel covariance matrix such that the mean of the eigen values equals to one (equal to the i.i.d. channel case $\mathbf{\Sigma}_{\mathrm{h}} = I_t$).

This performance metric was also used in [12] and [19].

B. Distortion Analysis of Optimal MISO Channel Quantizers

By employing the general framework described in Section II, the finite-rate quantized MISO beamforming system can be formulated as a general fixed-rate vector quantization problem [18] [28]. Specifically, the source variable to be quantized is denoted as $\overline{\mathbf{v}} = \left[\mathbf{v}_{\mathrm{R}}^{\mathsf{T}},\, \mathbf{v}_{\mathrm{I}}^{\mathsf{T}}\right]^{\mathsf{T}}$ of $2t$ real dimensions with $\mathbf{v}_{\mathrm{R}}$ and $\mathbf{v}_{\mathrm{I}}$ representing the real and imaginary parts of the complex channel directional vector $\mathbf{v}$. The encoder side information is denoted as $\alpha = \|\mathbf{h}\|^2$ of dimension $k_\alpha = 1$ representing the squared norm of the vector channel. For vectors in the vicinity of $\widehat{\mathbf{v}}$ (with $\widehat{\mathbf{v}}_{\mathrm{R}}$ and $\widehat{\mathbf{v}}_{\mathrm{I}}$ representing its real and imaginary parts), source variable $\overline{\mathbf{v}}$ is restricted under the constraint function given by

$$\mathbf{g}(\mathbf{v}) = \begin{bmatrix} \mathbf{v}_{\mathrm{R}}^{\mathsf{T}}\mathbf{v}_{\mathrm{R}} + \mathbf{v}_{\mathrm{I}}^{\mathsf{T}}\mathbf{v}_{\mathrm{I}} - 1 \\ \mathbf{v}_{\mathrm{R}}^{\mathsf{T}}\widehat{\mathbf{v}}_{\mathrm{I}} - \mathbf{v}_{\mathrm{I}}^{\mathsf{T}}\widehat{\mathbf{v}}_{\mathrm{R}} \end{bmatrix} = 0 \;, \tag{25}$$

where the first element represents the norm constraint $\|\mathbf{v}\| = 1$, and the second element represents the phase constraint $\measuredangle\langle \mathbf{v}, \widehat{\mathbf{v}}\rangle = 0$. Function $\mathbf{g}(\mathbf{v})$ has size $k_{\mathrm{c}} = 2$, which leads to the actual degrees of freedom of the quantization variable $\mathbf{v}$ to be $k_{\mathrm{q}}' = 2t - 2$. The instantaneous capacity loss due to effects of finite-rate CSI quantization is taken to be the system distortion function $D_{\mathrm{Q}}(\mathbf{v},\, \widehat{\mathbf{v}}\,;\, \alpha)$, which is given by the following form according to the definition given by (24)

$$D_{\mathrm{Q}}(\mathbf{v},\, \widehat{\mathbf{v}}\,;\, \alpha) = C_{\mathrm{L}}(\mathbf{h}, \widehat{\mathbf{v}}) \stackrel{\triangle}{=} -\log_2\left(1 - \frac{\rho\alpha}{1+\rho\alpha}\cdot\left(1 - \left|\langle \mathbf{v},\, \widehat{\mathbf{v}}\rangle\right|^2\right)\right) \;. \tag{26}$$

where α is the instantaneous channel power given by $\alpha = \|\mathbf{h}\|^2$.

By further utilizing the distortion analysis described in Section II, the optimal normalized inertial profile of a MISO system is tightly lower bounded by

$$\widetilde{I}_{\text{c, opt}}\left(\widehat{\mathbf{v}}\,;\, \alpha\right) = \frac{(t-1)\cdot \gamma_t^{-\frac{1}{t-1}}\cdot \rho\,\alpha}{\ln 2\cdot t\cdot (1+\rho\,\alpha)}\,, \qquad\qquad \gamma_t = \frac{\pi^{t-1}}{(t-1)!} \;. \tag{27}$$

And the average system distortion (or capacity loss) of a CSI-quantized MISO system can be lower bounded by

$$\widetilde{D}_{\text{c-Low}}\left(\boldsymbol{\Sigma}_{\mathrm{h}}\right) = \left(\frac{(t-1)\,\gamma_t^{-\frac{t}{t-1}}\cdot\rho\cdot\beta_1\left(\rho,\, t,\, \boldsymbol{\Sigma}_{\mathrm{h}}\right)}{\ln 2\cdot|\boldsymbol{\Sigma}_{\mathrm{h}}|}\right)\cdot 2^{-\frac{B}{t-1}} \;, \tag{28}$$

where $\beta_1\left(\rho,\, t,\, \boldsymbol{\Sigma}_{\mathrm{h}}\right)$ is a constant coefficient that only depends on the number of antennas t, channel correlation matrix $\boldsymbol{\Sigma}_{\mathrm{h}}$ and system SNR ρ, and is given by

$$\beta_1\left(\rho,\, t,\, \boldsymbol{\Sigma}_{\mathrm{h}}\right) = \left(\int_{\mathbf{v}:\,\mathbf{g}(\mathbf{v})=0}\left(\left(\mathbf{v}^{\mathsf{H}}\boldsymbol{\Sigma}_{\mathrm{h}}^{-1}\,\mathbf{v}\right)^{-(t+1)}\cdot {}_2F_0\left(t+1,\, 1;\,;\, -\frac{\rho}{\mathbf{v}^{\mathsf{H}}\boldsymbol{\Sigma}_{\mathrm{h}}^{-1}\,\mathbf{v}}\right)\right)^{\frac{t-1}{t}} d\,\mathbf{v}\right)^{\frac{t}{t-1}} \;, \tag{29}$$

with ${}_2F_0(\,;\,;\,)$ representing the generalized hypergeometric function. The optimal point density $\lambda^*(\mathbf{v})$ that achieves the minimal distortion is given by

$$\lambda^*(\mathbf{v}) = \beta_1(\rho,\, t,\, \mathbf{\Sigma}_{\mathrm{h}})^{-\frac{t-1}{t}} \cdot \left(\left(\mathbf{v}^{\mathsf{H}} \mathbf{\Sigma}_{\mathrm{h}}^{-1}\, \mathbf{v}\right)^{-(t+1)} \cdot {}_2F_0\left(t+1,\, 1;\, ;\, -\frac{\rho}{\mathbf{v}^{\mathsf{H}}\mathbf{\Sigma}_{\mathrm{h}}^{-1}\,\mathbf{v}}\right)\right)^{\frac{t-1}{t}} \quad . \tag{30}$$

As a special case, when the fading channel responses are spatially uncorrelated, i.e. $\mathbf{\Sigma}_{\mathrm{h}} = I_t$, the average system distortion has the following form

$$\widetilde{D}_{\text{c-Low}} = \left(\frac{t-1}{\ln 2} \cdot {}_2F_0\big(t+1,\, 1;\, ;\, -\rho\big) \cdot \rho\right) \cdot 2^{-\frac{B}{t-1}} \quad , \tag{31}$$

with the optimal point density $\lambda^*(\mathbf{v})$ being a uniform distribution given by

$$\lambda^*(\mathbf{v}) = \gamma_t^{-1}\,, \qquad \mathbf{v} \in \left\{\mathbf{v} \,\middle|\, \mathbf{g}(\mathbf{v}) = 0\right\} \quad . \tag{32}$$

Note that the derivations of the distortion (or capacity loss) analysis for the finite-rate CSI-quantized MISO beamforming systems provided in this section is brief due to the space limit. Please refer to [19] and [27] for more details.

V. Analysis of MISO CSI Quantizers with Transformed Codebook

In practical situations, the spatial correlation conditions of the fading channel responses may change during the transmission process. However, for a real system, it is impossible to design different codebooks optimized for every instantiation of the channel covariance matrix and it might also be infeasible for the transmitter and receiver to store a large amount of codebooks and use them adaptively. In these cases, it is convenient to use a channel quantizer whose codebook is generated from a fixed pre-designed codebook through a transformation parameterized by the channel covariance matrix. By utilizing the distortion analysis of the transformed codebooks provided in Section III, this section provides an investigation of the capacity loss of a finite-rate CSI-quantized MISO beamforming system over spatially correlated fading channels that uses transformed CSI quantizers.

A. Problem of MISO Channel Quantizers with Transformed Codebook

To be specific, suppose $\mathcal{C}_0$ is the optimal codebook designed for the i.i.d. MISO fading channels. When the elements of the fading channel response $\mathbf{h}$ are correlated, i.e. $\mathbf{h} \sim \mathcal{N}_{\mathrm{c}}(\mathbf{\Sigma}_{\mathrm{h}})$, it is evident that codebook $\mathcal{C}_0$ is no longer optimal. In order to compensate the mismatch between $\mathcal{C}_0$ and the current channel statistics, a transformed codebook $\mathcal{C}$ can be generated by the following manner,

$$\mathcal{C} = \left\{\, \mathbf{F}\big(\widehat{\mathbf{v}}\big) \;\middle|\; \widehat{\mathbf{v}} \in \mathcal{C}_0 \,\right\} \quad , \tag{33}$$

where $\mathbf{F}(\cdot)$ is a general non-linear transformation that depends on the channel statistics. Optimization of the transformation $\mathbf{F}(\cdot)$ turns out to be difficult, and hence a simple sub-optimal transformation,

$$\mathbf{F}(\widehat{\mathbf{v}}) = \frac{\mathbf{G}\,\widehat{\mathbf{v}}}{\|\mathbf{G}\,\widehat{\mathbf{v}}\|} \quad , \tag{34}$$

was proposed in [6] [14] where $\mathbf{G} \in \mathbb{C}^{t\times t}$ is a fixed matrix which depends on the channel covariance matrix $\mathbf{\Sigma}_{\text{h}}$. In the next subsection, distortion analysis of CSI-quantizers with transformed codebooks is provided.

B. Distortion Analysis of Quantizers with Transformed Codebook

First, according to the codebook transformation given by (34), the transformed point density function $\lambda_{\text{c-tr}}(\mathbf{v})$, from equation (12), has the following form

$$\lambda_{\text{c-tr}}(\mathbf{v}) = \gamma_t^{-1} \cdot |\mathbf{\Sigma}|^{-1} \cdot \left(\mathbf{v}^{\mathsf{H}}\mathbf{\Sigma}^{-1}\,\mathbf{v}\right)^{-t} \quad , \qquad \mathbf{\Sigma} = \mathbf{G}\cdot\mathbf{G}^{\mathsf{H}} \ . \tag{35}$$

which is equivalent to the PDF of a unit-norm complex vector $\mathbf{x}/\|\mathbf{x}\|$ with $\mathbf{x}$ having complex Gaussian distribution $\mathbf{x} \sim \mathcal{N}_{\text{c}}(\mathbf{0},\,\mathbf{\Sigma})$. It is evident that the transformed point density given by (35) does not match the optimal point density function $\lambda^*(\mathbf{v})$ given by (30) in the general case. However, for MISO systems with a large number of antennas and in high-SNR and low-SNR regimes, it can be shown that the optimal point density $\lambda^*(\mathbf{v})$ reduces to be the source distribution $p_{\mathbf{v}}(\mathbf{x})$ given by the following form

$$\lim_{t\to\infty} \ \lambda^*(\mathbf{x}) = p_{\mathbf{v}}\left(\mathbf{x}\right) = \gamma_t^{-1} \cdot |\mathbf{\Sigma}_{\text{h}}|^{-1} \cdot \left(\mathbf{x}^{\mathsf{H}}\mathbf{\Sigma}_{\text{h}}^{-1}\,\mathbf{x}\right)^{-t} \ . \tag{36}$$

In this case, by choosing matrix $\mathbf{G}$ as $\mathbf{G} = \mathbf{U}\,\mathbf{\Lambda}^{\frac{1}{2}}$ with matrices $\mathbf{U}$ and $\mathbf{\Lambda}$ obtained from the eigen-value decomposition of the channel covariance matrix, i.e. $\mathbf{\Sigma}_{\text{h}} = \mathbf{U}\,\mathbf{\Lambda}\,\mathbf{U}^{\mathsf{H}}$, one can generate a transformed codebook $\mathcal{C}$ whose point density $\lambda_{\text{c-tr}}(\mathbf{v})$ is equal to the optimal point density function $\lambda^*(\mathbf{v})$. Under this codebook transformation (when t is large), there is no distortion loss caused by the point density mismatch, though the system suffers from the oblongitis of the Voronoi shape.

By substituting the transformation given by (34) into equation (17), the inertial profile of the transformed codebook with sub-optimal encoder $\mathcal{Q}_{\text{sub}}$ (or encoding process) is given by

$$\widetilde{I}_{\text{c-sub}}\left(\mathbf{v}\,;\,\alpha\right) = \frac{\gamma_t^{-\frac{1}{t-1}} \cdot \rho\alpha \cdot \left(\mathbf{v}^{\mathsf{H}}\,\mathbf{\Sigma}^{-1}\,\mathbf{v}\right)}{t\cdot\ln 2\cdot(1+\rho\alpha)} \cdot \mathbf{tr}\Bigg(\left(I - \mathbf{v}\,\mathbf{v}^{\mathsf{H}}\right)\cdot\mathbf{\Sigma}\Bigg) \geq \widetilde{I}_{\text{c-opt}}\left(\mathbf{v}\,;\,\alpha\right) \ . \tag{37}$$

where $\widetilde{I}_{\text{c-opt}}\left(\mathbf{v}\,;\,\alpha\right)$ is the optimal inertia profile given by equation (27). It is evident from (37) that except for unitary rotations of the i.i.d. codebook, any non-trivial transformation of codebook $\mathcal{C}_0$ will lead to mismatched Voronoi shapes and hence causes inertial profile loss. Therefore, a codebook transformation that makes the best compromise between the point density loss and the inertial profile loss is favored.

Finding the optimal codebook transformation $\mathbf{F}$ that minimizes the system distortion turns out to be a difficult problem. In this paper, instead of optimizing the overall distortion w.r.t. matrix $\mathbf{G}$, we provide a distortion analysis of MISO systems with transformed CSI-quantizers using codebooks generated by the heuristic choice $\mathbf{\Sigma}_{\mathrm{h}} = \mathbf{G} \cdot \mathbf{G}^{\mathsf{H}}$ (or[2] $\mathbf{G} = \mathbf{U}\,\mathbf{\Lambda}^{\frac{1}{2}}$). To be specific, by substituting the transformed point density (35) and the transformed inertia profile (37) into the distortion integral given by (19), the corresponding upper and lower bounds of the average system distortion of a MISO CSI-quantizer with transformed codebook has the following forms

$$\widetilde{D}_{\text{c-tr-Low}} = \frac{(t-1)\cdot|\mathbf{\Sigma}_{\mathrm{h}}|^{\frac{1}{t-1}}}{\ln 2\cdot t}\cdot E\left[\frac{\rho\cdot\left(\mathbf{h}^{\mathsf{H}}\mathbf{\Sigma}_{\mathrm{h}}^{-1}\mathbf{h}\right)^{\frac{t}{t-1}}}{\left(1+\rho\cdot\|\mathbf{h}\|^2\right)\cdot\|\mathbf{h}\|^{\frac{2}{t-1}}}\right]\cdot 2^{-\frac{B}{t-1}}\ , \tag{38}$$

$$\widetilde{D}_{\text{c-tr-Upp}} = \frac{|\mathbf{\Sigma}_{\mathrm{h}}|^{\frac{1}{t-1}}}{\ln 2\cdot t}\cdot E\left[\frac{\rho\cdot\left(\mathbf{h}^{\mathsf{H}}\mathbf{\Sigma}_{\mathrm{h}}^{-1}\mathbf{h}\right)^{\frac{2t-1}{t-1}}\cdot\left(t\cdot\|\mathbf{h}\|^2-\mathbf{h}^{\mathsf{H}}\mathbf{\Sigma}_{\mathrm{h}}\,\mathbf{h}\right)}{\left(1+\rho\cdot\|\mathbf{h}\|^2\right)\cdot\|\mathbf{h}\|^{\frac{4t-2}{t-1}}}\right]\cdot 2^{-\frac{B}{t-1}}\ . \tag{39}$$

Some numerical experiments were conducted to get a better feel for the utility of the bounds. Fig. 1 shows the system capacity loss due to the finite-rate quantization of the CSI versus feedback rate B for a 3×1 MISO system over correlated Rayleigh fading channels under different system SNRs at $\rho = -10$ and 20 dB, respectively. The spatially correlated channel is simulated by the correlation model in [29]: A linear antenna array with antenna spacing of half wavelength, i.e. $D/\lambda = 0.5$, uniform angular-spread in $[-30^\circ, 30^\circ]$ and angle of arrival $\phi = 0^\circ$. Simulation results of both the optimal designed codebook using the minimal mean-squared weighted inner product (MSwIP) criterion proposed in [11] as well as the sub-optimal transformed codebook are plotted. For comparison purpose, distortion lower bound $\widetilde{D}_{\text{c-tr-Low}}$ given by (38) and the distortion upper bound $\widetilde{D}_{\text{c-tr-Upp}}$ given by (39) are also included in the plot. It can be observed from Fig. 1 that the distortion lower bound $\widetilde{D}_{\text{c-tr-Low}}$ is tight and the performance of the CSI quantizer with transformed codebook is close to that of the optimal codebooks.

C. Performance Comparison with Optimal MISO CSI-Quantizers

In order to assess the sub-optimality caused by codebook transformation, one would like to compare the system performance in terms of the average distortion of quantizers using transformed codebooks with that of the optimally designed codebooks. Interestingly, in high-SNR and low-SNR regimes with a large number transmit antennas t, the average system distortion of CSI quantizers with transformed

[2] Note that the codebook transformation is not unique. Any right unitary rotation $\mathbf{G}\cdot\mathbf{P}$ on matrix $\mathbf{G}$, with $\mathbf{P}\cdot\mathbf{P}^{\mathrm{H}} = I$, can generate another codebook transformation (or codebook) with the same performance.

codebook can be upper and lower bounded by some multiplicative factors of the distortion of optimal quantizers.

Proposition 2: For MISO systems with a large number of transmit antennas, i.e. $t \to \infty$, the following inequalities are satisfied

$$\widetilde{D}_{\text{c-Low}}^{\text{H-dim, H-SNR}} \overset{a}{=} \widetilde{D}_{\text{c-tr-Low}}^{\text{H-dim, H-SNR}} \leq \widetilde{D}_{\text{c-tr}}^{\text{H-dim, H-SNR}} \leq \widetilde{D}_{\text{c-tr-Upp}}^{\text{H-dim, H-SNR}} \overset{b}{\leq} c_1 \cdot \widetilde{D}_{\text{c-Low,1}}^{\text{H-dim, H-SNR}} \;, \tag{40}$$

$$\widetilde{D}_{\text{c-Low}}^{\text{H-dim, L-SNR}} \overset{a}{=} \widetilde{D}_{\text{c-tr-Low}}^{\text{H-dim, L-SNR}} \leq \widetilde{D}_{\text{c-tr}}^{\text{H-dim, L-SNR}} \leq \widetilde{D}_{\text{c-tr-Upp}}^{\text{H-dim, L-SNR}} \overset{b}{\leq} c_2 \cdot \widetilde{D}_{\text{c-Low,1}}^{\text{H-dim, L-SNR}} \;, \tag{41}$$

where the superscript "H-dim" represents the high-dimensional distortion (t large), and "H-SNR" (or "L-SNR") represents the distortion in high-SNR (or low-SNR) regimes. In equation (40), the constant coefficients c_1 and c_2 are given by the following form

$$c_1 = \left(\frac{\delta(t-2)}{\lambda_{\text{h},1} \cdot \lambda_{\text{h},2}} - (t-1)(t-2) \sum_{i=1}^{t} \frac{\left(\ln \lambda_{\text{h},i}\right)/\lambda_{\text{h},i}^2}{\prod_{k \neq i} \left(1 - \lambda_{\text{h},k}/\lambda_{\text{h},i}\right)} \right) \Big/ c_1^{\frac{t}{t-1}} \;, \tag{42}$$

$$c_2 = (t-1) \sum_{i=1}^{t} \frac{\left(\ln \lambda_{\text{h},i}\right)/\lambda_{\text{h},i}}{\prod_{k \neq i} \left(1 - \lambda_{\text{h},k}/\lambda_{\text{h},i}\right)} \;. \tag{43}$$

Proof: See Appendix B. ■

Note from proposition 2 that constants c_1 and c_2 can be viewed as the upper bounds of the penalty paid for using a transformed codebook instead of the optimal design. Based on the numerical examples, c_1 and c_2 are slightly greater than 1 for most channels that are not "highly" correlated. This means that the intuitive choice of $\mathbf{F}$ given in [6] [14] is a fairly good solution especially for cases when the channel covariance matrix has a relative small condition number.

We plot in Fig. 2 the normalized capacity loss, defined to be the distortion ratio of correlated fading channels to that of the distortion of i.i.d. fading channels, for a 3×1 MISO system as a function of antenna spacing D/λ with both optimal and transformed codebooks. The average system signal to noise ratio is $\rho = -10$ dB, and the quantization resolution is $B = 10$ bits per channel update. For comparison purpose, the ratio of the distortion bounds, i.e. $\widetilde{D}_{\text{c-tr-Low}}(\mathbf{\Sigma}_\text{h})/\widetilde{D}_{\text{c-tr-Low}}(I_t)$ and $\widetilde{D}_{\text{c-tr-Upp}}(\mathbf{\Sigma}_\text{h})/\widetilde{D}_{\text{c-tr-Upp}}(I_t)$, are also included in the plot. It can be observed from Fig. 2 that the analytical bounds agree well with the obtained simulation results.

In order to demonstrates the tightness of the distortion bounds $\widetilde{D}_{\text{c-tr-Upp}}$ and $\widetilde{D}_{\text{c-tr-Low}}$ in high-SNR and low-SNR regimes, Fig. 3 plots the constant coefficient c_1 and c_2 versus the number of transmit antennas t for correlated MISO channels with adjacent antenna spacing $D/\lambda = 0.5$. From the plot, it can be observed that the performance degradation caused by the transformed codebook is less than 10% in low-SNR regimes and 22% in high-SNR regimes for MISO systems with more than 10 transmit antennas.

VI. Conclusion

This paper extends the high resolution quantization theory approach to study the effects of a finite-rate MISO CSI-quantizer using transformed codebook transmitting over correlated fading channels. The contributions of this paper are twofold. First, analysis is provided for a generalized vector quantizer with transformed codebook. Bounds on the average system distortion of this class of quantizers were provided. It exposes the effects of two kinds sub-optimality, which include the sub-optimal point density loss and the mismatched Voronoi shape oblongness. Second, we focused our attention on the application of the proposed general framework to provide the capacity analysis of a feedback-based MISO system over correlated fading channels using channel quantizers with transformed codebooks. In particular, upper and lower bounds on the channel capacity loss of MISO systems with transformed codebooks were provided and compared to that of the optimal quantizers. It was further proved that the average distortion of CSI quantizers with transformed codebooks can be upper and lower bounded by some multiplicative factors of the distortion of optimal quantizers. This factor was shown to be close to one for fading channels whose channel covariance matrix has small to moderate condition numbers. Numerical and simulation results were presented which confirm the tightness of the theoretical distortion bounds.

VII. Acknowledgement

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Appendix A: Proof of Proposition 1

Proof: Due to the fixed location of the code points in the transformed codebook $\mathcal{C}$, which can not be optimized to minimize the normalized inertial profile, it is evident that the transformed inertial profile $\widetilde{I}_{\text{tr}}$ is lower bounded by the optimal inertial profile $\widetilde{I}_{\text{opt}}$ given by equation (5). Hence, inequality (a) in (16) can be obtained after some manipulations. The same reasonings are valid for inequality (a) in (17) for the constraint source.

As for inequality (b) in (16), since function $\mathbf{F}(\cdot)$ is first order continuous, any points in the vicinity of the transformed code point $\mathbf{F}(\widehat{\mathbf{y}})$ has a first order Taylor series expansion given by

$$\mathbf{F}(\mathbf{y}) \approx \mathbf{F}(\widehat{\mathbf{y}}) + \mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}}) \cdot (\mathbf{y} - \widehat{\mathbf{y}}) \,, \qquad \mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}}) = \frac{\partial}{\partial \mathbf{y}}\bigg|_{\mathbf{y}=\widehat{\mathbf{y}}} \mathbf{F}(\mathbf{y}) \ . \tag{44}$$

Moreover, due to the fact the $\mathbf{F}(\cdot)$ is a one-to-one mapping, for any point $\mathbf{y}'$ in the vicinity of $\mathbf{F}(\widehat{\mathbf{y}})$, there exists a unique point $\mathbf{y}$ in the neighborhood of $\widehat{\mathbf{y}}$ such that $\mathbf{y}' = \mathbf{F}(\mathbf{y})$. Therefore, under high resolutions, the distortion function D_{Q} can be expanded around point $\mathbf{F}(\widehat{\mathbf{y}})$ as follows

$$\begin{aligned} D_{\mathrm{Q}}\big(\mathbf{y}', \mathbf{F}(\widehat{\mathbf{y}}) \,;\, \mathbf{z}\big) \;&\approx\; \big(\mathbf{y}' - \mathbf{F}(\widehat{\mathbf{y}})\big)^{\mathsf{T}} \mathbf{W}_{\mathbf{z}}\big(\mathbf{F}(\widehat{\mathbf{y}})\big) \big(\mathbf{y}' - \mathbf{F}(\widehat{\mathbf{y}})\big) \\ &\approx\; (\mathbf{y} - \widehat{\mathbf{y}})^{\mathsf{T}} \cdot \Big(\mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}})^{\mathsf{T}} \cdot \mathbf{W}_{\mathbf{z}}\big(\mathbf{F}(\widehat{\mathbf{y}})\big) \cdot \mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}})\Big) \cdot (\mathbf{y} - \widehat{\mathbf{y}}) \ , \end{aligned} \tag{45}$$

which has quadratic form but with transformed sensitivity matrix. By substituting equation (45) as well as the Voronoi shape of the sub-optimal encoder given by equation (15) into the definition of the inertial profile given by (3), we can obtain the following normalized inertial profile of the transformed code with sub-optimal encoder,

$$\begin{aligned} &\widetilde{I}_{\mathrm{tr}}\left(\mathbf{F}(\widehat{\mathbf{y}}) \,;\, \mathbf{z}\right) \le \widetilde{I}_{\mathrm{sub}}\left(\mathbf{F}(\widehat{\mathbf{y}}) \,;\, \mathbf{z}\right) \\ &\quad = \frac{\big|\mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}})\big|^{-2/k_{\mathrm{q}}}}{k_{\mathrm{q}}+2} \cdot \left(\frac{\big|\mathbf{W}_{0,\mathbf{z}}(\widehat{\mathbf{y}})\big|}{\kappa_{k_{\mathrm{q}}}^{2}}\right)^{\frac{1}{k_{\mathrm{q}}}} \cdot \mathbf{tr}\Big(\mathbf{W}_{0,\mathbf{z}}(\widehat{\mathbf{y}})^{-1} \cdot \mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}})^{\mathsf{T}} \cdot \mathbf{W}_{\mathbf{z}}\big(\mathbf{F}(\widehat{\mathbf{y}})\big) \cdot \mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}})\Big) \ , \end{aligned} \tag{46}$$

which corresponds to inequality (b) in (16).

If the source variable (vector) $\mathbf{y}$ is further subject to k_{c} *constraints* given by the vector equation $\mathbf{g}(\mathbf{y}) = \mathbf{0}$, the distortion function D_{Q} can be similarly expanded around point $\mathbf{F}(\widehat{\mathbf{y}})$ as

$$\begin{aligned} D_{\mathrm{Q}}\big(\mathbf{y}', \mathbf{F}(\widehat{\mathbf{y}}) \,;\, \mathbf{z}\big) \;&\approx\; (\mathbf{y} - \widehat{\mathbf{y}})^{\mathsf{T}} \cdot \Big(\mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}})^{\mathsf{T}} \cdot \mathbf{W}_{\mathbf{z}}\big(\mathbf{F}(\widehat{\mathbf{y}})\big) \cdot \mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}})\Big) \cdot (\mathbf{y} - \widehat{\mathbf{y}}) \\ &=\; \mathbf{e}^{\mathsf{T}} \cdot \Big(\mathbf{V}_2(\widehat{\mathbf{y}})^{\mathsf{T}} \cdot \mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}})^{\mathsf{T}} \cdot \mathbf{W}_{\mathbf{z}}\big(\mathbf{F}(\widehat{\mathbf{y}})\big) \cdot \mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}}) \cdot \mathbf{V}_2(\widehat{\mathbf{y}})\Big) \cdot \mathbf{e} \ , \end{aligned} \tag{47}$$

where $\mathbf{e}$ is the projected error vector with respect to point $\widehat{\mathbf{y}}$ given by

$$\mathbf{e} = \mathbf{V}_2(\widehat{\mathbf{y}})^{\mathsf{T}} \cdot (\mathbf{y} - \widehat{\mathbf{y}}) \ . \tag{48}$$

Similarly, by substituting (47) and the sub-optimal Voronoi shape (15) into the inertial profile definition (3), we can obtain the sub-optimal inertial profile of the transformed code with constraint source

$$\begin{aligned} &\widetilde{I}_{\text{tr-c}}\left(\mathbf{F}(\widehat{\mathbf{y}}) \,;\, \mathbf{z}\right) \le \widetilde{I}_{\text{sub-c}}\left(\mathbf{F}(\widehat{\mathbf{y}}) \,;\, \mathbf{z}\right) = \frac{\Big|\mathbf{V}_2(\widehat{\mathbf{y}})^{\mathsf{T}} \cdot \mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}}) \cdot \mathbf{V}_2(\widehat{\mathbf{y}})\Big|^{-\frac{2}{k'_{\mathrm{q}}}}}{k'_{\mathrm{q}}+2} \left(\frac{\Big|\mathbf{V}_2(\widehat{\mathbf{y}})^{\mathsf{T}} \cdot \mathbf{W}_{0,\mathbf{z}}(\widehat{\mathbf{y}}) \cdot \mathbf{V}_2(\widehat{\mathbf{y}})\Big|}{\kappa_{k_{\mathrm{q}}}^{2}}\right)^{\frac{1}{k_{\mathrm{q}}}} \\ &\quad \cdot \mathbf{tr}\left(\Big(\mathbf{V}_2(\widehat{\mathbf{y}})^{\mathsf{T}} \cdot \mathbf{W}_{0,\mathbf{z}}(\widehat{\mathbf{y}} \cdot \mathbf{V}_2(\widehat{\mathbf{y}})\Big)^{-1} \cdot \mathbf{V}_2(\widehat{\mathbf{y}})^{\mathsf{T}} \cdot \mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}})^{\mathsf{T}} \cdot \mathbf{W}_{\mathbf{z}}\big(\mathbf{F}(\widehat{\mathbf{y}})\big) \cdot \mathbf{F}_{\mathrm{d}}(\widehat{\mathbf{y}}) \cdot \mathbf{V}_2(\widehat{\mathbf{y}})\right) \ , \end{aligned} \tag{49}$$

which corresponds to inequality (b) in (17). ■

Appendix B: Proof of Proposition 2

Proof: First, in high-SNR regimes, distortion bounds $\widetilde{D}^{\text{H-dim, H-SNR}}_{\text{c-tr-Low}}$ and $\widetilde{D}^{\text{H-dim, H-SNR}}_{\text{c-tr-Upp}}$ can be represented as

$$\widetilde{D}^{\text{H-dim, H-SNR}}_{\text{c-tr-Low}} = \left(\frac{(t-1)\cdot|\boldsymbol{\Sigma}_{\text{h}}|^{\frac{1}{t-1}}\cdot\beta_2}{\ln 2\cdot t}\right)\cdot 2^{-\frac{B}{t-1}} \approx \widetilde{D}^{\text{H-dim, H-SNR}}_{\text{c-Low,1}}\ , \tag{50}$$

$$\widetilde{D}^{\text{H-dim, L-SNR}}_{\text{c-tr-Upp}} \leq \left(\frac{(t-1)\cdot|\boldsymbol{\Sigma}_{\text{h}}|^{\frac{1}{t-1}}\cdot\beta_3}{\ln 2\cdot t}\right)\cdot 2^{-\frac{B}{t-1}} = \left(\beta_3\cdot\beta_2^{-\frac{t}{t-1}}\right)\cdot\widetilde{D}^{\text{H-dim, H-SNR}}_{\text{c-Low,1}}\ , \tag{51}$$

where coefficients β_2 and β_3 can be expressed as the expected powers of the ratios of Gaussian quadratic variables, which are given by

$$\beta_2 = E\left[\frac{\mathbf{h}^{\mathsf{H}}\boldsymbol{\Sigma}_{\text{h}}^{-1}\,\mathbf{h}}{\mathbf{h}^{\mathsf{H}}\,\mathbf{h}}\right]\ , \qquad \beta_3 = E\left[\left(\frac{\mathbf{h}^{\mathsf{H}}\boldsymbol{\Sigma}_{\text{h}}^{-1}\,\mathbf{h}}{\mathbf{h}^{\mathsf{H}}\,\mathbf{h}}\right)^2\right]\ . \tag{52}$$

The moments of ratios of random variables, including central quadratic forms in normal variables, were investigated in [30], and the results can described by the following integrals

$$E\left[\left(\frac{X}{Y}\right)^n\right] = \Gamma(n)^{-1}\int_0^\infty v^{n-1}M^{(n)}_{X,Y}(0,-v)\ d\,v\ , \tag{53}$$

where $M_{X,Y}(u,v)$ is the joint moment generating function (m.g.f.) of random variables X and Y, and $M^{(n)}_{X,Y}(0,-v)$ stands for $\partial^n M_{X,Y}(u,-v)/\partial v^n$ evaluated at $u=0$. Therefore, by setting $X = \mathbf{h}^{\mathsf{H}}\boldsymbol{\Sigma}_{\text{h}}^{-1}\,\mathbf{h}$ and $Y = \mathbf{h}^{\mathsf{H}}\,\mathbf{h}$, the joint m.g.f. of variables X and Y can be represented as

$$M_{X,Y}(u,v) = \frac{1}{\mathbf{det}\Big(I-\big(u\cdot I+v\cdot\boldsymbol{\Sigma}_{\text{h}}\big)\Big)} = \left(\prod_{k=1}^{t}\left(1-u-v\cdot\lambda_{\text{h},k}\right)\right)^{-1}\ . \tag{54}$$

By substituting the joint m.g.f. given by equation (54) into the integration (53) with $n=1$, coefficient β_2, after some manipulations, has the following closed-form expression

$$\beta_2 = (t-1)\sum_{i=1}^{t}\frac{\big(\ln\lambda_{\text{h},\,i}\big)/\lambda_{\text{h},\,i}}{\prod_{k\neq i}\big(1-\lambda_{\text{h},\,k}/\lambda_{\text{h},\,i}\big)}\ . \tag{55}$$

Finally, by substituting (55) into equation (50), equality (a) of (40) is proved. With similar reasoning, by substituting the joint m.g.f. (54) into equation (53) with $n=2$, coefficient β_3 is obtained. Correspondingly, a closed-form expression of coefficient $c_1 = \beta_3\cdot\beta_2^{-\frac{t}{t-1}}$, given by equation (42), can also be obtained, and inequality (b) of (40) is proved.

Similarly, in Low-SNR regimes, distortion bounds $\widetilde{D}^{\text{H-dim, L-SNR}}_{\text{c-tr-Low}}$ and $\widetilde{D}^{\text{H-dim, L-SNR}}_{\text{c-tr-Upp}}$ has the following forms

$$\widetilde{D}^{\text{H-dim, L-SNR}}_{\text{c-tr-Low}} = \left(\frac{(t-1)\cdot|\boldsymbol{\Sigma}_{\text{h}}|^{\frac{1}{t-1}}\cdot\beta_4\cdot\rho}{\ln 2}\right)\cdot 2^{-\frac{B}{t-1}} = \beta_4\cdot\widetilde{D}^{\text{H-dim, L-SNR}}_{\text{c-Low,1}}\ , \tag{56}$$

$$\widetilde{D}^{\text{H-dim, L-SNR}}_{\text{c-tr-Upp}} \leq \left(\frac{(t-1)\cdot|\boldsymbol{\Sigma}_{\text{h}}|^{\frac{1}{t-1}}\cdot\beta_5\cdot\rho}{\ln 2}\right)\cdot 2^{-\frac{B}{t-1}} = \beta_5\cdot\widetilde{D}^{\text{H-dim, L-SNR}}_{\text{c-Low,1}}\ , \tag{57}$$

where the coefficients β_4 and β_5 are given by

$$\beta_4 = E\left[\frac{\mathbf{h}^{\mathsf{H}}\mathbf{\Sigma}_{\mathrm{h}}^{-1}\,\mathbf{h}}{t}\right], \qquad\qquad \beta_5 = E\left[\frac{\left(\mathbf{h}^{\mathsf{H}}\mathbf{\Sigma}_{\mathrm{h}}^{-1}\,\mathbf{h}\right)^2}{t\cdot\mathbf{h}^{\mathsf{H}}\,\mathbf{h}}\right]\ . \tag{58}$$

From equation (58), it is evident that $\beta_4 = 1$, and hence the equality (a) of equation (41) can be proved. Moreover, by extending the results of the moments of the quadratic forms provided in [30], the following expectation can be obtained after some manipulations

$$E\left[\frac{X^2}{Y}\right] = \int_0^{\infty} \frac{\partial^2 M_{X,Y}(u,-v)}{\partial\, u^2}\bigg|_{u=0} d\,v\ , \tag{59}$$

Therefore, by setting $X = \mathbf{h}^{\mathsf{H}}\mathbf{\Sigma}_{\mathrm{h}}^{-1}\,\mathbf{h}$ and $Y = \mathbf{h}^{\mathsf{H}}\,\mathbf{h}$, and substituting the joint m.g.f. given by equation (54) into the integration (59), coefficient β_5 can be obtained. It is equivalent to coefficient c_2 given by equation (43), and hence the inequality (b) of (41) can be proved.

■

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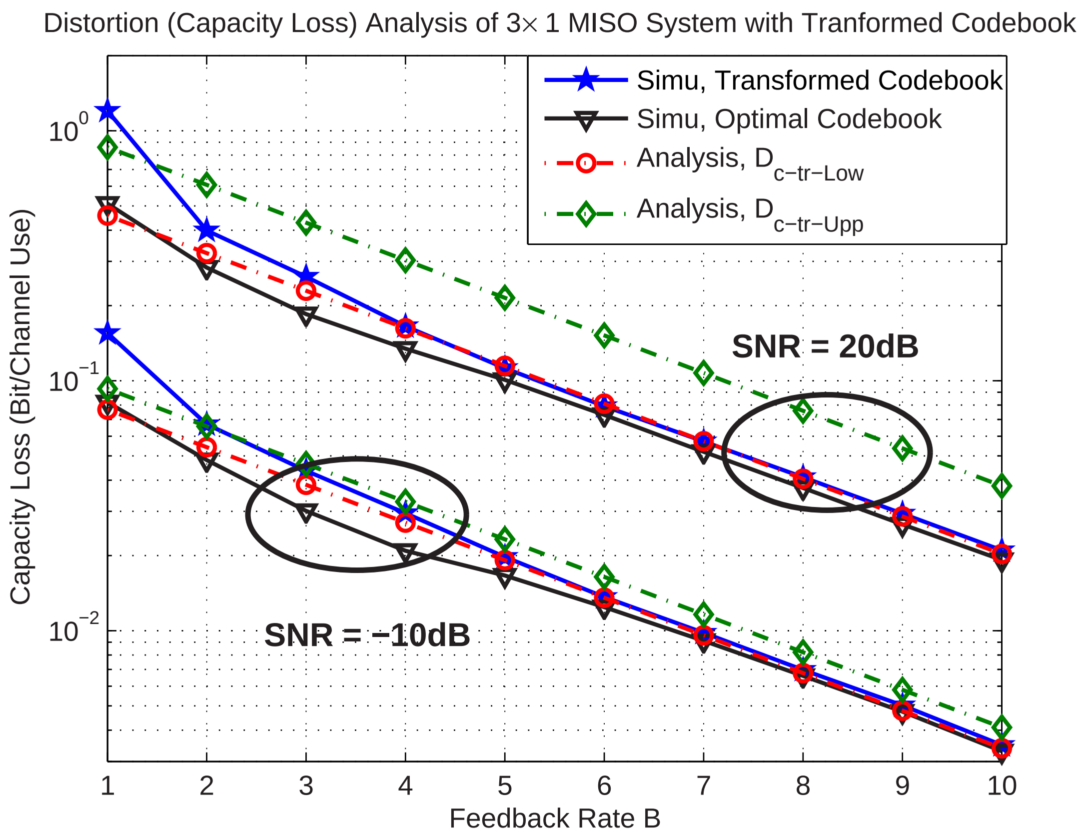

Fig. 1. Capacity loss of a 3×1 correlated MISO system with normalized antenna spacing $d = D/\lambda = 0.5$ versus CSI feedback rate B using different channel quantization codebooks (Optimal codebook vs Transformed codebook).

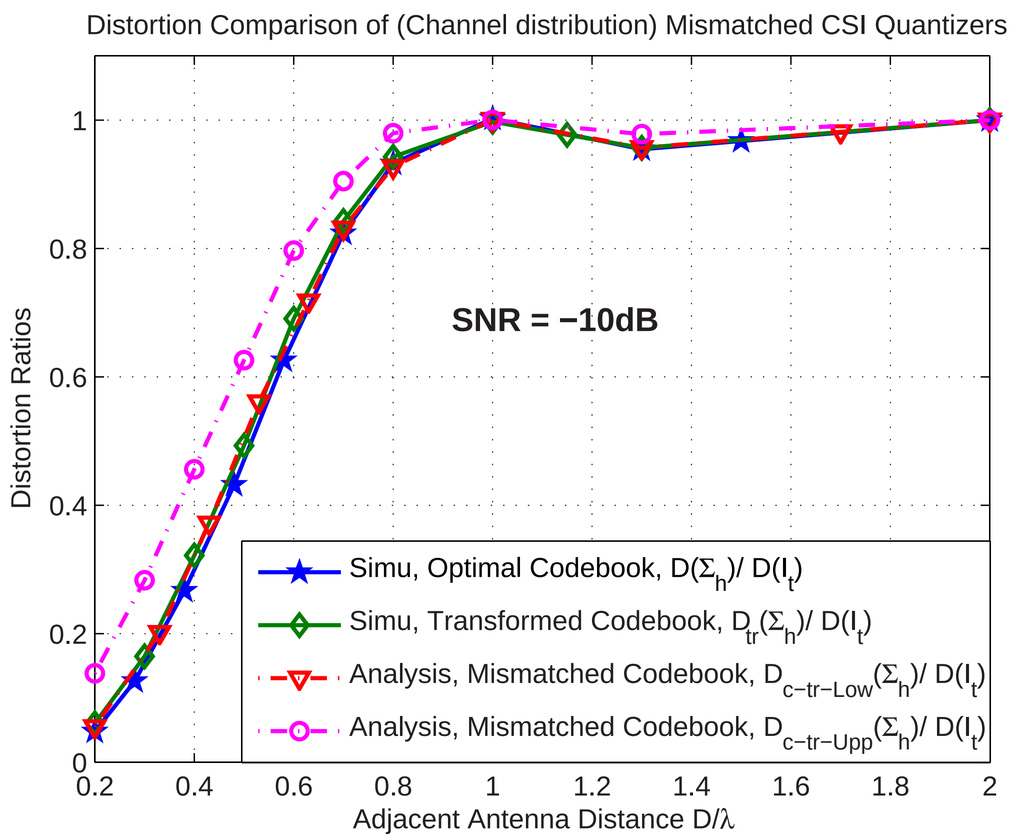

Fig. 2. Normalized capacity loss (w.r.t. the capacity loss of uncorrelated fading channels) comparison of a 3×1 MISO transmit beamforming with optimal and transformed codebooks versus antenna spacing $d = D/\lambda$, in low SNR regimes ($\rho = -10$ dB).

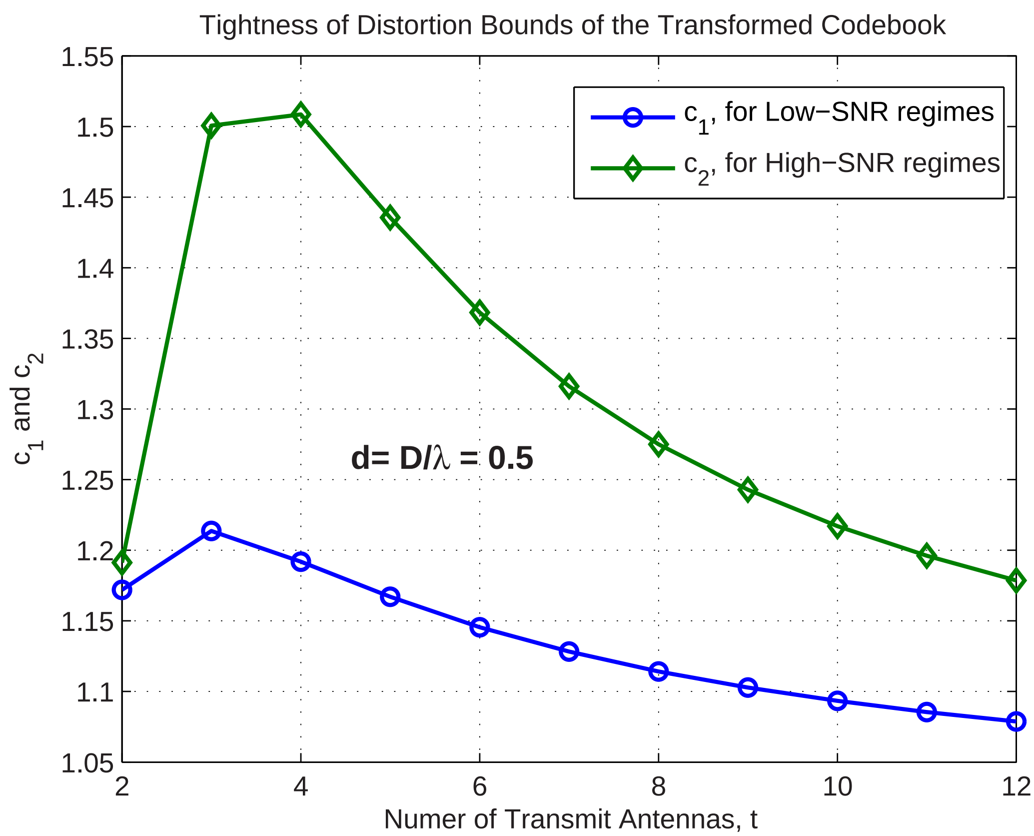

Fig. 3. Demonstration of the tightness of the distortion bounds $D_{\text{c-tr-Low}}$ and $D_{\text{c-tr-Upp}}$ for a MISO system using transformed codebook over correlated fading channels with different number of transmit antennas of antenna spacing $d = D/\lambda = 0.5$.